Spherical Derivative of Meromorphic Function with Image of Finite Spherical Area

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Let $\Omega$ be a domain in the complex plane $\mathbb{C}$ with the Poincaré metric $P_\Omega(z)|dz|$ which is $|dz|/(1 - |z|^2)$ if $\Omega$ is the open unit disk. Suppose that the Riemann sphere $\mathbb{C} \cup \{\infty\}$ of radius $1/2$, so that it has the area $\pi$ and let $0 < \beta < \pi$. Let $\alpha_{\Omega,\beta}(z), z \in \Omega$, be the supremum of the spherical derivative $|f'(z)|(1 + |f(z)|^2)$ of $f$ meromorphic in $\Omega$ such that the spherical area of the image $f(\Omega) \subset \mathbb{C} \cup \{\infty\}$ is not greater than $\beta$. Then

$$\alpha_{\Omega,\beta}(z) \leq \sqrt{\frac{\beta}{\pi - \beta}} P_\Omega(z), \quad z \in \Omega.$$ 

The equality holds if and only if $\Omega$ is simply connected.

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1. INTRODUCTION

The complex plane $\mathbb{C} = \mathbb{R}^2$, together with the point $\infty$ at infinity, is identified with the Riemann sphere $\mathbb{C}^*$ of radius $1/2$ touching $\mathbb{C}$ from above at the origin with the aid of the stereographic projection viewed from the north pole of $\mathbb{C}^*$. The sphere $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is the metric subspace of the Euclidean space $\mathbb{R}^3$, so that it has the distance $X(z, w)$

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which can be expressed by

\[ X(z, w) = \frac{|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}, \quad z, w \in \mathbb{C}; \]

\[ X(z, \infty) = X(\infty, z) = \frac{1}{\sqrt{1 + |z|^2}}, \quad z \in \mathbb{C}; \quad X(\infty, \infty) = 0. \]

The spherical area of a set \( E \subseteq \mathbb{C}^* \) is given by the double integral

\[ A(E) = \iint_{E \setminus \{\infty\}} \frac{dx \, dy}{(1 + |z|^2)^2}, \quad z = x + iy; \]

here \( E \) is identified with its projection. For example, the spherical cap \( E(a, r) = \{ z \in \mathbb{C}^*; X(z, a) < r \} \) of center \( a \in \mathbb{C}^* \) and radius, \( r, 0 < r \leq 1 \), has the area \( \pi r^2 \), so that \( A(\mathbb{C}) = A(E(0, 1)) = \pi \). Actually, \( A(E(a, r)) = A(E(0, r)) \).

Let \( \mathcal{M}(\Omega) \) be the family of all the meromorphic functions in a domain \( \Omega \subseteq \mathbb{C} \); the constant function \( \infty \) is regarded as a member of \( \mathcal{M}(\Omega) \). The spherical derivative of \( f \in \mathcal{M}(\Omega) \) at \( z \in \Omega \) is defined by

\[ f^#(z) = \lim_{|w - z| \to 0} \frac{X(f(w), f(z))}{|w - z|}; \]

hence \( f^#(z) = |f'(z)|/(1 + |f(z)|^2) \) if \( f(z) \neq \infty \) and \( f^#(z) = |(1/f)'(z)| \) if \( f(z) = \infty \). Note that \( (\infty)^#(z) \equiv 0 \). The set of \( w \in \mathbb{C}^* \) assumed by \( f \) at least once in \( \Omega \) is denoted by \( f(\Omega) \); hence \( f(\Omega) \subseteq \mathbb{C}^* \) and \( w \in f(\Omega) \) if and only if \( w = f(z) \) for a \( z \in \Omega \). For a constant \( \beta, 0 < \beta < \pi \), we let \( \mathcal{F}(\Omega, \beta) \) be the set of all \( f \in \mathcal{M}(\Omega) \) such that \( A(f(\Omega)) \leq \beta \). Note that \( A(f(\Omega)) \leq \iint_{\Omega} f^#(z)^2 \, dx \, dy \); the right-hand side integral may possibly be \( +\infty \).

We suppose that \( \Omega \) is hyperbolic, namely, \( \mathbb{C} \setminus \Omega \) contains at least two points. Let \( \phi \) be a universal covering projection from \( D = \{ w; |w| < 1 \} \) onto \( \Omega \), \( \phi \in \text{Proj}(\Omega) \) in notation; \( \phi \) is holomorphic and \( \phi' \) is zero-free. The Poincaré density \( P_{\Omega} \) is then the function in \( \Omega \) defined by

\[ P_{\Omega}(z) = \frac{1}{(1 - |w|^2)|\phi'(w)|}, \quad z \in \Omega, \]
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where $z = \phi(w)$; the choice of $\phi \in \text{Proj}(\Omega)$ and $w$ is immaterial as far as $z = \phi(w)$ is satisfied.

We begin with

**THEOREM 1** For each $f \in \mathcal{F}(\Omega, \beta)$ for hyperbolic $\Omega$, the estimate holds:

$$f^\#(z) \leq \sqrt{\frac{\beta}{\pi - \beta}} P_\Omega(z) \quad (1.1)$$

at each $z \in \Omega$. If the equality holds in (1.1) at a point $z \in \Omega$, then $f$ maps $\Omega$ univalently onto a spherical cap. Conversely if $\Omega$ is mapped by a meromorphic function $f \in \mathcal{F}(\Omega, \beta)$ univalently onto a spherical cap, then there exists exactly one point $z \in \Omega$ where the equality holds in (1.1).

**THEOREM 2** The family $\mathcal{F}(\Omega, \beta)$ for hyperbolic $\Omega$ is compact. Namely, given $f_n \in \mathcal{F}(\Omega, \beta), n = 1, 2, \ldots$, we have a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ and an $f \in \mathcal{F}(\Omega, \beta)$ such that

$$\max_{z \in E} X(f_{n_j}(z), f(z)) \to 0 \quad \text{as } n_j \to \infty \quad (1.2)$$

for each compact set $E$ (in $\mathbb{C}$) comprised in $\Omega$.

Set

$$c_{\Omega, \beta}(z) = \sup \{f^\#(z); f \in \mathcal{F}(\Omega, \beta)\}, \quad z \in \Omega.$$ 

It then follows from Theorem 2 that the supremum is the maximum; $f^\#(z) = c_{\Omega, \beta}(z)$ for an $f \in \mathcal{F}(\Omega, \beta)$. For this extremal function we set

$$g(w) = \begin{cases} 
\frac{f(w) - f(z)}{e^{i \arg f'(z)}(1 + f(z)f(w))}, & \text{if } f(z) \neq \infty; \\
1, & \text{if } f(z) = \infty.
\end{cases}$$

Then $g \in \mathcal{F}(\Omega, \beta)$ with $g(z) = 0$ and $0 \leq g'(z) = c_{\Omega, \beta}(z)$. Again, $g$ is extremal.

**THEOREM 3** If $\Omega$ is hyperbolic, then

$$c_{\Omega, \beta}(z) \leq \sqrt{\frac{\beta}{\pi - \beta}} P_\Omega(z) \quad (1.3)$$
at each \( z \in \Omega \). If the equality holds in (1.3) at a point \( z \in \Omega \), then \( \Omega \) is simply connected. If \( \Omega \) is simply connected, then the equality holds in (1.3) everywhere in \( \Omega \).

We omit the detailed proof of Theorem 3 because we have only to apply Theorem 2 to an extremal function.

2. PROOF OF THEOREM 1

**Lemma 1**  For \( f \in \mathcal{F}(D, \beta) \) the inequality holds:

\[
\left( f'(0) \right)^2 \leq \frac{\beta}{\pi - \beta}.
\]

The equality holds in (2.1) if and only if

\[
f(z) = \frac{az + b}{1 - baz}, \quad z \in D,
\]

where \( a \in \mathbb{C} \) and \( b \in \mathbb{C}^* \) are constants with

\[
|a| = \sqrt{\frac{\beta}{\pi - \beta}}.
\]

Read \( f(z) = -1/(az) \) in case \( b = \infty \).

This lemma is Dufresnoy's, the case \( r_0 = 1 \) in [2, Lemma I, (2)]. The equality condition in the present “if and only if” form is obtained in the similar manner as in [1, pp. 219–220]. Note that Dufresnoy adopted the unit sphere of center at the origin in \( \mathbb{R}^3 \) as the Riemann sphere, so that we need obvious changes.

**Proof of Theorem 1**  We choose \( \phi \in \text{Proj}(\Omega) \) with \( z = \phi(0) \) and we observe that \( f \circ \phi \in \mathcal{F}(D, \beta) \). Since

\[
(f \circ \phi)'(0) = f'(z)|\phi'(0)| = f'(z)/P_\Omega(z),
\]
the inequality (1.1) is a consequence of (2.1). The equality holds in (1.1) at z if and only if

\[(f \circ \phi)(w) = \frac{aw + b}{1 - bw}, \quad w \in D,\]

where \(|a| = \sqrt{\beta}/(\pi - \beta)\). Hence \(f\) is univalent in \(\Omega\), so that \(f \circ \phi\) is univalent in \(D\). The image \(f(\Omega)\) is \(\{(aw + b)/(1 - bw); w \in D\}\) which is the image of the cap \(E(0, |a|/\sqrt{(1 + |a|^2)}\) by the rotation of \(C^*\): \(T(\zeta) = (\zeta + b)/(1 - b\zeta)\), so that \(f(\Omega)\) is a spherical cap.

Suppose that \(f \in \mathcal{F}(\Omega, \beta)\) maps \(\Omega\) univalently onto a spherical cap \(E(a, \sqrt{\beta}/\pi), a \in C^*\) and set \(\rho = \sqrt{\beta}/(\pi - \beta)\). Then

\[
\phi(w) = f^{-1}\left(\frac{\rho w + a}{1 - \rho w}\right)
\]

is in \(\text{Proj}(\Omega)\). Since

\[
\frac{f^\#(\phi(w))}{P_\Omega(\phi(w))} = \frac{\rho(1 - |w|^2)}{1 + \rho^2|w|^2},
\]

it follows that \(f^\#(\phi(w)) = \rho P_\Omega(\phi(w))\) if and only if \(w = 0\). Hence the equality holds in (1.1) at exactly one point \(\phi(0)\), the inverse of \(a\) by \(f\).

If \(z \neq w\), then

\[
X(z, w) \leq \arctan \left| \frac{z - w}{1 + \overline{w}z} \right| = \int_\Gamma dX(\zeta),
\]

where \(dX(\zeta) = |d\zeta|/(1 + |\zeta|)\) and \(\Gamma\) is the projection of the shorter of the great circle passing through, and bisected by \(z, w\); in case \(z = -1\) we have many \(\Gamma\). Here \(0 < \arctan \rho < \pi/2\) for \(0 < \rho \leq +\infty\).

Suppose that \(\beta = A(\Omega) < \pi\) for a domain \(\Omega \subset C\). We then have

\[
dX(z) \leq \sqrt{\frac{\beta}{\pi - \beta}} P_\Omega(z)|dz|, \quad z \in \Omega.
\]

More precisely,

\[
\frac{1}{1 + |z|^2} \leq \sqrt{\frac{A(\Omega)}{\pi - A(\Omega)}} P_\Omega(z), \quad z \in \Omega. \tag{2.3}
\]
The equality holds in (2.3) at a point $z \in \Omega$ if and only if $\Omega$ is a spherical cap
\[ \left\{ \frac{a\zeta + z}{1 - \bar{a}\zeta}, \zeta \in D \right\}, \]
where $a \in \mathbb{C}$ and $|a| = \sqrt{A(\Omega)/(\pi - A(\Omega))}$. For the proof we have only to follow that of Theorem 1 with $f(\zeta) \equiv \zeta, f \in \mathcal{F}(\Omega, \beta)$.

3. PROOF OF THEOREM 2

We begin with the case $\Omega = D$. Since $f^\#$ of $f \in \mathcal{F}(D, \beta)$ is bounded by
\[ \sqrt{\frac{\beta}{\pi - \beta}} \cdot \frac{1}{1 - r^2} \]
on $\{|z| < r\}, 0 < r < 1$, it follows that $\mathcal{F}(D, \beta)$ is equicontinuous in $X$ on $\{|z| < r\}$. Hence $\mathcal{F}(D, \beta)$ is normal in that sense that given $\{f_n\} \subset \mathcal{F}(D, \beta)$, we have a subsequence $\{f_{n_k}\} \subset \{f_n\}$ and $f \in \mathcal{M}(D)$ such that (1.2) holds for each compact set $E$ comprised in $D$. To prove that $f \in \mathcal{F}(D, \beta)$ we may suppose that $f$ is nonconstant.

For simplicity we suppose that
\[ \text{max}_{z \in E} X(f_n(z), f(z)) \to 0 \quad \text{as } n \to \infty \]
for $f_n \in \mathcal{F}(D, \beta), n = 1, 2, \ldots$ We shall then prove that for each $b \in D$ we have $r = r(b) > 0$ and a natural number $N = N(b)$ such that $f(\Delta) \subset f_n(\Omega)$ for all $n > N$, where
\[ \Delta = \{z; |z - b| < r\} \subset D. \]

Then for each compact set $K \subset D$ we have $n$ such that $f(K) \subset f_n(\Omega)$, so that $A(f(K)) \leq \beta$, whence $f \in \mathcal{F}(\Omega, \beta)$.

We first suppose that $f(b) \neq \infty$. Then we have $r_1 > 0$ and a constant $M_1 > 0$ such that $f$ is pole-free and bounded, $|f| < M_1$, on $\Delta_1 = \{z; |z - b| \leq r_1\} \subset D$. We then find a constant $M_2 \geq M_1$ and a natural number $N_1$ such that $f_n$ is pole-free and bounded, $|f_n| < M_2$, on $\Delta_1$ for
$n \geq N_1$. Hence

$$\max_{z \in \Delta_1} |f_n(z) - f(z)| \to 0 \quad \text{as} \quad n \to \infty. \quad (3.2)$$

Consequently, for $\Delta$ of (3.1) with $0 < r < r_1$, we have

$$\sup_{z \in \Delta} |f'(z) - f'(z)| \to 0 \quad \text{as} \quad n \to \infty. \quad (3.3)$$

To prove that the present $\Delta$ is the requested we suppose that there exists $q \in \Delta$ such that $p = f(q) \notin f_n(\Omega)$ for infinitely many $n_j, n_1 < n_2 < \cdots$. Choose $R > 0$ such that

$$\{z; |z - q| \leq R\} \subset \Delta \quad \text{and} \quad p \notin f(\{z; 0 < |z - q| \leq R\}),$$

and set

$$c = \{z; |z - q| = R\}.$$

The argument principle then shows that

$$1 \leq \frac{1}{2\pi i} \int_c \frac{f''(\zeta)}{f(\zeta) - p} \, d\zeta.$$

The right-hand side integral is, with the aid of (3.2) and (3.3), the limit

$$\lim_{n_j \to \infty} \frac{1}{2\pi i} \int_c \frac{f_n''(\zeta)}{f_n(\zeta) - f_n(q)} \, d\zeta = 0.$$

This is a contradiction.

In the case $f(b) = \infty$ we consider $\{1/f_n\} \subset F(\Omega, \beta)$ with $1/f \in M(\Omega)$ and arrive at a contradiction again.

For general $\Omega$ we fix $\phi \in \text{Proj}(\Omega)$. Then, for each compact set $E$ in $\Omega$ we may find a compact set $E_1 \subset D$ such that $\phi(E_1) = E$. Furthermore, $\phi$ is automorphic with respect to the universal covering transformation group $G: \phi = \phi \circ T, T \in G$. Since $f_n \circ \phi \in F(D, \beta)$, we have a subsequence $\{f_n\}$ of $\{f_n\}$ and $g \in F(D, \beta)$ such that $f_n \circ \phi$ converges to $g$ on each compact set in $D$. Since $g$ is then automorphic with respect to $G$, we have $f \in M(\Omega)$ such that $f \circ \phi = g$ and this $f$ is the requested.
4. CONFORMAL INVARIANT \( c_{\Omega, \beta}(z) \)

Let \( \Sigma \) be another domain in \( \mathbb{C} \) and let \( f \) be holomorphic in \( \Omega \) with \( f(\Omega) \subset \Sigma \). Then

\[
c_{\Sigma, \beta}(f(z))|f'(z)| \leq c_{\Omega, \beta}(z), \quad z \in \Omega.
\]

In particular, if \( f \) is univalent and \( f(\Omega) = \Sigma \), then

\[
c_{\Sigma, \beta}(f(z))|f'(z)| = c_{\Omega, \beta}(z), \quad z \in \Omega,
\]

so that \( c_{\Omega, \beta}(z)|dz| \) is conformally invariant.

Let \( B_z(\Omega) \) be the family of all \( f \) holomorphic, bounded, \( |f| < 1 \), in \( \Omega \), and further, \( f(z) = 0, z \in \Omega \). Then

\[
\gamma_{\Omega}(z) = \sup\{|f'(z)|; f \in B_z(\Omega)\}
\]

is called the analytic capacity of \( \Omega \) at \( z \). Then \( \gamma_{\Omega}(z) \) is the maximum \( \gamma_{\Omega}(z) = |f'(z)| = f'(z) \) for a unique \( f \in B_z(\Omega) \) called the Ahlfors function of \( \Omega \) at \( z \). See [3, p. 110]. Since \( B_z(\Omega) \subset F(\Omega, \pi/2) \), it follows that

\[
\gamma_{\Omega}(z) \leq c_{\Omega, \pi/2}(z), \quad z \in \Omega.
\]

On the other hand, it follows from (1.3) that

\[
c_{\Omega, \pi/2}(z) \leq P_{\Omega}(z), \quad z \in \Omega.
\]

If \( f \in F(\Omega, \beta) \), then \( C^* \setminus f(\Omega) \) is of positive spherical area, so that this set is of positive capacity. Hence \( f \) is of uniformly bounded characteristic in \( \Omega \); see [5, Theorem 1] and also [4, 6]. Suppose that each function of uniformly bounded characteristic in \( \Omega \) is constant. Then \( f \in F(\Omega, \beta) \) is a constant.

References

