On a Generalization of the Osgood Condition

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In this paper a generalization of the famous uniqueness Osgood condition is given. This new result is important for many applications.

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1. INTRODUCTION

We consider nonlinear Volterra equations of the following type:

\[ u(x) = \int_0^x (x - s)^{a-1} g(u(s)) \, ds \quad (x \geq 0, \alpha \geq 1), \]

(1.1)

where the kernel \( k \) and the nonlinearity \( g \) are nonnegative. Moreover \( g(u) = 0 \) for \( u \leq 0 \).

This type of equation appears in some applications such as nonlinear diffusion problems or shock wave propagation [1]. It is clear that \( u(x) \equiv 0 \) is the trivial solution of (1.1) but from the physical point of view only nonnegative solutions of the considered equation are interesting.

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This problem is a very special case of the problem of the uniqueness of the trivial solution of the equation

\[ u(x) = \int_0^x k(x, s, u(s)) \, ds \quad (x \geq 0). \]

If the trivial solution is unique one says that \( k \) is a Kamke function and this question appears in many problems not directly connected with the uniqueness of the solution [2]. In this paper we will consider only \( k(x, s, u) = (x - s)^\alpha g(u) \). If we put \( \alpha = 1 \) in (1.1), then the uniqueness of the trivial solution is equivalent to the uniqueness of the trivial solution to the problem: \( u' = g(u), \ u(0) = 0 \). If \( g \) is a nondecreasing continuous function \( (g(0) = 0) \), then the uniqueness answer is given by

\[ \int_0^\delta \frac{ds}{g(s)} = \infty. \]

If the last integral is finite, the problem \( u' = g(u), \ u(0) = 0 \) has a nontrivial solution.

Having in mind the physical applications of (1.1), different mathematicians since the eighties have tried to generalize the Osgood condition for (1.1). It has been shown [1,3–6] that for a nondecreasing continuous \( g \) \( (g(0) = 0) \) the trivial solution is unique for (1.1) if and only if

\[ \int_0^\delta \frac{ds}{\phi_0(s)} = \infty, \quad \text{where} \quad \phi_0(s) = s \left[ \frac{g(s)}{s} \right]^{1/\alpha}. \quad (1.2) \]

Let us note that for \( \alpha = 1 \) we obtain the classical Osgood condition. But in some applications [7,8] there appear nonlinearities \( g \) which behave like \( u^p \) \( (p \in (-1, 0)) \). In this case the generalized Osgood condition does not work. In recent papers [9,10] a new condition for the uniqueness of the trivial solution in the case of \( g \) not necessarily increasing has been presented. But this was done for an integer \( \alpha \geq 2 \). In this note we want to present the generalization of the condition (1.2) for all the \( \alpha > 1 \) and nonlinearities \( g \) general enough.

We assume

(i) \( g(s) \) is continuous for \( s > 0 \) and \( g(s)s^{1/(\alpha - 1)} \to 0 \) as \( s \to 0^+; \)

(ii) there exists \( m \geq 0 \) such that \( g(s)s^m \) is nondecreasing in the right-hand side vicinity of zero.
Now we can formulate

**Theorem** Let \( a > 1 \) and let \( g \) satisfy (i) and (ii). Then the trivial solution \( u(x) = 0 \) is unique if and only if

\[
\int_0^6 \frac{ds}{\phi(s)} = \infty, \quad \text{where} \quad \phi(s) = s^{(a-2)/(a-1)}[\psi(s)]^{1/\alpha} \tag{1.3}
\]

and

\[
\psi(s) = s^{2-\alpha} \int_0^s (s-t)^{a-2}g(t)t^{-(a-2)/(a-1)} dt. \tag{1.4}
\]

**Remark 1.1** We shall prove theorem in the following equivalent form:

Equation (1.1) has a nontrivial solution, i.e. a continuous function \( u \) such that \( u(x) > 0 \) for \( x > 0 \), if and only if

\[
\int_0^6 \frac{ds}{\phi(s)} < \infty.
\]

**Remark 1.2** If \( g \) is a nondecreasing continuous function, then an easy comparison of \( \phi \) with \( s(g(s)/s)^{1/\alpha} \) shows that the conditions (1.2) and (1.3) are equivalent.

**Remark 1.3** One can check easily that in the case \( g(u) = u^{-\beta}, \beta \geq 1/(\alpha - 1) \) Eq. (1.1) only has the trivial solution. Because of this we assume in (i) that \( \lim_{s \to 0^+} g(s)s^{1/(\alpha - 1)} = 0 \) If (1.1) has a nontrivial solution, then the condition \( \lim_{s \to 0^+} g(s)s^{1/(\alpha - 1)} = 0 \) is equivalent to the following one \( \int_0^s g(s)s^{-(2)/(\alpha - 1)}ds < \infty \). It is also known [10] that the last condition is necessary for the existence of nontrivial solutions of (1.1) in the case \( \alpha \geq 2 \). The case \( \alpha \in (1,2) \) is still open.

**Remark 1.4** Slight modifications of assumptions (i) and (ii) allow us also to consider \( g \) which behave at the origin like \( |\sin(1/x)| \) [10].

**2. Main Steps of the Proof of the Theorem**

The proof of the theorem is based mainly on some *a priori* estimates of nontrivial solutions and properties of auxiliary functions. Since similar
arguments to those used in [10] apply to the case $\alpha \geq 2$, we concentrate on
$\alpha \in (1, 2)$. As in [11] we can show

**Lemma 2.1** Let $\mu$ be a Borel measure on $[0, a]$ ($a > 0$). Then the function

$$ u(x) = \int_0^x (x - s)^\beta \, d\mu(s) \quad (\beta > 0) $$

is absolutely continuous and there exists constants $c_1, c_2 > 0$ such that

$$ c_1 u'(x)^\beta \leq \int_0^x (u(x) - u(s))^{\beta - 1} \, d\mu(s) \leq c_2 u'(x)^\beta $$

for $x \in [0, a]$.

**Remark 2.1** The function $x^{-\beta} u(x)$ is nondecreasing.

**Lemma 2.2** Let $\alpha > 1$. Then the nontrivial solution of (1.1) is increasing
and there exist constants $c_1, c_2 > 0$ such that

$$ c_1 v(x)^{\alpha - 1} \leq \int_0^x (x - s)^{\alpha - 2} g(s)[v(s)]^{-1} \, ds \leq c_2 v(x)^{\alpha - 1}, \quad (2.1) $$

where $v(x) = u'(u^{-1}(x))$.

To prove Lemma 2.2 we apply the results of Lemma 2.1 to (1.1) with
$\beta = \alpha - 1$ and $d\mu(s) = g(u(s)) \, ds$.

Throughout, a function $f: [0, a] \to [0, \infty)$ for which there exists a constant $c > 0$ such that

$$ f(x) \leq cf(y) \quad \text{for } 0 < x < y \leq a $$

will be called an almost monotonous function.

**Lemma 2.3** Let $\alpha \in (1, 2)$. Then the function $\psi$ defined by (1.4) is almost
monotonous.

**Proof of Lemma 2.3** First we note that

$$ \psi(s) = \int_0^s (s - t)^{\alpha - 2}[(s - t) + t]^{2 - \alpha} \psi_1(t) \, dt, $$

where $\psi_1(s) = g(s)s^{-(\alpha - 2)/(\alpha - 1)}$. 
We introduce the following auxiliary functions:

\[
\psi_2(s) = \int_0^s \psi_1(t) \, dt + \int_0^s (s - t)^{\alpha - 2} t^{1 - \alpha} \psi_1(t) \, dt,
\]

\[
\psi_3(s) = \psi_1(s)s + m \int_0^s \psi_1(t) \, dt,
\]

where \( m \) is given by (ii) and

\[
\psi_4(s) = \int_0^s \psi_1(t) \, dt + \int_0^s (s - t)^{\alpha - 2} t^{1 - \alpha} \psi_3(t) \, dt.
\]

Making the following observations

\[
\psi_3(s) = \lim_{\delta \to 0^+} \int_\delta^s t^{-m} \, d(t^{m+1} \psi_1(t))
\]

and

\[
\int_0^s (s - t)^{\alpha - 2} t^{1 - \alpha} \psi_3(t) \, dt = \int_0^1 (1 - t)^{\alpha - 2} t^{1 - \alpha} \psi_3(st) \, dt,
\]

we infer that the functions \( \psi_3 \) and \( \psi_4 \) are nondecreasing. Furthermore, we note that

\[
\psi_2(s) \leq \psi_4(s) \leq \max(\gamma, 1 + \gamma m) \psi_2(s) \quad (s \in (0, a]),
\]

where \( \gamma = \int_0^a (s - t)^{\alpha - 2} t^{1 - \alpha} \, dt \). Thus \( \psi_2 \) is almost monotonous.

Finally, we easily see that

\[
c_1 \psi_2(s) \leq \psi(s) \leq c_2 \psi_2(s) \quad (s \in (0, a])
\]

for some constants \( c_1, c_2 > 0 \), which gives our assertion.

Now we can prove the lemma:

**Lemma 2.4** Let \( \phi \) be given by (1.3) and \( u \) be a nontrivial solution to (1.1). Then there exist constants \( c_1, c_2 > 0 \) such that

\[
c_1 \phi(x) \leq v(x) \leq c_2 \phi(x) \quad (x \in (0, a]), \tag{2.2}
\]

where \( v(x) = u'(u^{-1}(x)) \).
Proof of Lemma 2.4 Let $\alpha \in (1, 2)$. We shall denote

$$h(x) = \int_0^x (x-s)^{a-2} g(s)[v(s)]^{-1}ds$$

and $h_1(x) = \int_0^x g(s)[v(s)]^{-1}ds$.

We have the following relations

$$h_1(x) = \text{const} \int_0^x (x-s)^{1-\alpha} h(s) ds$$

and

$$h(x) = \int_0^x (x-s)^{a-2} h_1'(s) ds.$$

By (2.1) we can write

$$\psi_1(s) = h_1'(s)(s^2-\alpha v(s)^{\alpha-1})^{1/(\alpha-1)}$$

$$\geq \text{const} h_1'(s)(s^{2-\alpha} h(s))^{1/(\alpha-1)}.$$  \hspace{1cm} (2.3)

Since

$$\omega(s; x) = \int_0^s (x-t)^{a-2} t^{2-\alpha} h_1'(t) dt \leq s^{2-\alpha} h(s) \quad (0 < s < x),$$

by (2.3) we get

$$h_1'(s)\omega(s; x)^{1/(\alpha-1)} \leq \text{const} \psi_1(s)$$  \hspace{1cm} (2.4)

for $s \in (0, x]$. We also have the inequality

$$\psi(x) = \int_0^x ((x-s) + s)^{2-\alpha} (x-s)^{a-2} \psi_1(s) ds$$

$$\geq \text{const} \int_0^x \psi_1(s) ds + \text{const} \int_0^x (x-s)^{a-2} s^{2-\alpha} \psi_1(s) ds$$

(the constants are positive). By (2.4) we can write

$$\psi(x) \geq \text{const} \int_0^x h_1'(s)h_1(s)^{1/(\alpha-1)} ds$$

$$\quad + \text{const} \int_0^x (x-s)^{a-2} s^{2-\alpha} h_1'(s)\omega(s; x)^{1/(\alpha-1)} ds.$$  \hspace{1cm} (2.5)

Since the last integral is equal to $\text{const} \left[\omega(x; x)\right]^{\alpha/(\alpha-1)}$, by (2.5) we get

$$\psi(x) \geq \text{const} (h_1(x) + \omega(x; x))^{\alpha/(\alpha-1)}.$$  \hspace{1cm} (2.6)
Noting that

\[ h_1(x) = \int_0^x (x - t)^{-2} (x - t)^{2-\alpha} h'(t) \, dt, \]

from (2.6) and the left-hand side of (2.1) we get

\[ \psi(x) \geq \text{const} [x^{2-\alpha} h(x)]^{\alpha/(\alpha-1)} \geq \text{const} x^{(2-\alpha)/(\alpha-1)} \, \nu(x)^\alpha. \]

Hence we obtain the right-hand side of (2.2) for \( \alpha \in (1, 2) \). By the right-hand side of (2.2) and the monotonous properties of \( \psi \) we have

\[ h(x) \geq \text{const} \int_0^x (x - s)^{-2} g(s) s^{-(\alpha-2)/(\alpha-1)} \, ds \, \psi(x)^{-1/\alpha}, \]

which gives

\[ h(x) \geq \text{const} x^{\alpha-2} [\psi(x)]^{(\alpha-1)/\alpha}. \tag{2.7} \]

From (2.7) and the right-hand side of (2.1) we get the left-hand side of (2.2) for \( \alpha \in (1, 2) \). The lemma is proved.

**Remark 2.2** If we consider the equation

\[ u_\epsilon(x) = \epsilon x^{\alpha-1} + \int_0^x (x - s)^{\alpha-1} g(u_\epsilon(s)) \, ds \quad (\alpha > 1) \tag{2.8} \]

then putting \( \mu(s) = \epsilon \delta_0 + g(u_\epsilon(s)) \, ds \) and repeating our considerations we have

\[ c_1 \left( \epsilon x^{\alpha-1} + \phi(x)^{\alpha-1} \right)^{1/(\alpha-1)} \leq v_\epsilon(x) \leq c_2 \left( \epsilon x^{\alpha-1} + \phi(x)^{\alpha-1} \right)^{1/(\alpha-1)}, \tag{2.9} \]

where \( c_1, c_2 > 0 \) and \( v_\epsilon(x) = u_\epsilon'(u_\epsilon^{-1}(x)) \).

**Sketch of the Proof of Theorem** If (1.1) has a nontrivial solution \( u \), then

\[ u^{-1}(x) = \int_0^x (u^{-1})'(s) \, ds = \int_0^x [\nu(s)]^{-1} \, ds. \]
By (2.2) we get
\[ \infty > u^{-1}(x) \geq \int_0^x [\phi(s)]^{-1} ds \]
and the necessary condition for the existence of nontrivial solutions is proved.

By Schauder-type arguments it can be shown that for every \( \epsilon \in (0, \epsilon_0) \) Eq. (2.8) has a nontrivial solution \( u_\epsilon \). Since all solutions satisfy (2.9), by the Arzela–Ascoli theorem [12] there exists a sequence \( \epsilon_n \to 0 \), as \( n \to \infty \) and the corresponding solutions \( u_n \) of (2.8) such that \( u_n(x) \) converges uniformly to a solution \( u(x) \) of (1.1) on the interval \([0, a]\) \( (a > 0) \) as \( n \to \infty \).

Since by (2.9)
\[ u_n^{-1}(x) \leq \text{const} \int_0^x \frac{ds}{\phi(s)} = F^{-1}(x), \]
or equivalently \( u_n(x) \geq F(x) \) on \([0, a]\) for all \( n \). This implies \( u(x) \geq F(x) \) on \([0, a]\) and \( u \) is a nontrivial solution to (1.1). Thus the sufficient condition for the existence of nontrivial solutions is proved.

References