On a Minimax Problem of Ricceri

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Let $E$ be a real separable and reflexive Banach space, $X \subseteq E$ weakly closed and unbounded, $\Phi$ and $\Psi$ two non-constant weakly sequentially lower semicontinuous functionals defined on $X$, such that $\Phi + \lambda \Psi$ is coercive for each $\lambda \geq 0$. In this setting, if

$$
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) + \rho)) = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\Psi(x) + \rho))
$$

for every $\rho \in \mathbb{R}$, then, one has

$$
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda \Psi(x) + h(\lambda)),
$$

for every concave function $h : [0, +\infty[ \to \mathbb{R}$.

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1. INTRODUCTION

Here and throughout the sequel, $E$ is a real separable and reflexive Banach space, $X$ is a weakly closed unbounded subset of $E$, and $\Phi, \Psi$ are two (non-constant) sequentially weakly lower semicontinuous functionals defined on $X$. For each $\lambda \geq 0$,

$$
\inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda \Psi(x) + h(\lambda)),
$$

for every concave function $h : [0, +\infty[ \to \mathbb{R}$.

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functionals on $X$ such that

$$\lim_{x \in X, \|x\| \to +\infty} (\Phi(x) + \lambda \Psi(x)) = +\infty$$

for all $\lambda \geq 0$.

In this setting, the importance of finding a continuous concave function $h : [0, +\infty[ \to \mathbb{R}$ such that

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda \Psi(x) + h(\lambda)),$$

has been clearly shown by Ricceri in a series of recent papers ([2–4]). Actually, if that happens, there is an open interval $I \subseteq ]0, +\infty[\ such that, for each $\lambda \in I$, the functional $\Phi + \lambda \Psi$ has a local non-absolute minimum in the relative weak topology of $X$. In turn, under further appropriate assumptions, this fact leads to a three critical points theorem (Theorem 1 of [4] improving Theorem 3.1 of [3]) which is a new, useful tool to get multiplicity results for non-linear boundary value problems ([1,2,4]).

In [3], just in view of an application to the Dirichlet problem, Ricceri pointed out a natural way to get (1), with a linear $h$ ([3], Proposition 3.1). At the same time, he asked ([3], Remark 5.2) whether it may happen that for a suitable continuous concave function $h$, (1) holds, while, for every $\rho \in \mathbb{R}$, one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda (\Psi(x) + \rho)) = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda (\Psi(x) + \rho)).$$

The aim of this paper is to answer, in negative, Ricceri's question.

Our result is as follows.

**Theorem 1**  Under the assumptions above, the following assertions are equivalent:

(i) For every $\rho \in \mathbb{R}$, one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda (\Psi(x) - \rho)) = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda (\Psi(x) - \rho)).$$
(ii) For every $\rho \in \inf \Psi$, $\sup x \Psi[\ , \ one \ has$

$$\sup_{x \in \Psi^{-1}([\rho, +\infty])} \frac{\Phi(x) - \inf_{\Psi^{-1}([\rho, +\infty])} \Phi}{\rho - \Psi(x)} \leq \inf_{x \in \Psi^{-1}([0, +\infty])} \frac{\Phi(x) - \inf_{\Psi^{-1}([0, +\infty])} \Phi}{\rho - \Psi(x)}.$$

(iii) For every concave function $h : [0, +\infty[ \rightarrow \mathbb{R}$, one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda \Psi(x) + h(\lambda)).$$

2. PRELIMINARY LEMMAS

The proof of Theorem 1 needs some rather delicate lemmas. We prove them in this section. From now on, we denote l.s.c. as lower semicontinuous, u.s.c. as upper semicontinuous, $\chi_{[a, b]}$ the characteristic function of a real interval $[a, b]$.

Except for Lemmas and 4, we also assume that the two functions $\Phi, \Psi$ satisfy (ii) of Theorem 1.

**Lemma 1** Assume that $\alpha, \beta \in \mathbb{R}^+$ with $\alpha < \beta$, then the following assertions are equivalent:

(i) For every $\rho \in \mathbb{R}$, one has

$$\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda (\Psi(x) - \rho)) = \inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda (\Psi(x) - \rho)).$$

(ii) For every $\rho \in \inf \Psi$, $\sup_x \Psi[\ , \ one \ of \ the \ following \ two \ pairs \ of \ inequalities \ holds:$

$$\sup_{x \in \Psi^{-1}([\rho, +\infty])} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} > \beta,$$

$$\sup_{x \in \Psi^{-1}([\rho, +\infty])} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)} \leq \inf_{x \in \Psi^{-1}([0, +\infty])} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)}.$$
or

\[
\sup_{x \in \Psi^{-1}([\rho, +\infty[)} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} \leq \beta, \tag{4}
\]

\[
\sup_{x \in \Psi^{-1}([\rho, +\infty[)} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} \leq \inf_{x \in \Psi^{-1}([\rho, +\infty[)} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} \tag{5}
\]

whereas

\[
a(\rho) = \inf_{x \in \Psi^{-1}([\rho, +\infty[)} (\Phi(x) + \alpha(\Psi(x) - \rho))
\]

and

\[
b(\rho) = \inf_{x \in \Psi^{-1}([\rho, +\infty[)} (\Phi(x) + \beta(\Psi(x) - \rho)).
\]

**Proof**

(iii) \(\Rightarrow\) (iiii)

Fix \(\rho \in \inf_X \Psi, \sup_X \Psi[,\) we have

\[
\inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \min\{a(\rho), b(\rho)\}.
\]

We prove that

\[
b(\rho) < a(\rho) \Rightarrow (2) \text{ and } (3)
\]

\[
a(\rho) \leq b(\rho) \Rightarrow (4) \text{ and } (5).
\]

Suppose \(b(\rho) < a(\rho),\) then \(\inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)) = b(\rho),\) moreover there exists \(\tilde{x} \in \Psi^{-1}([\rho, +\infty[)\) such that \(\Phi(\tilde{x}) + \beta(\Psi(\tilde{x}) - \rho) < a(\rho),\) then

\[
\frac{\Phi(\tilde{x}) - a(\rho)}{\rho - \Psi(\tilde{x})} > \beta
\]

that implies (2).

Since \(\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = b(\rho),\) there exists \(\lambda_\rho \in [\alpha, \beta]\) such that

\[
\inf_{x \in X} (\Phi(x) + \lambda_\rho(\Psi(x) - \rho)) = b(\rho),
\]
thus we have

\[
\sup_{x \in \Psi^{-1}(\rho, \infty]} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)} \leq \lambda_\rho \leq \inf_{x \in \Psi^{-1}(-\infty, \rho]} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)},
\]

that implies (3).

Suppose \(a(\rho) \leq b(\rho)\), then \(\inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)) = a(\rho)\).

The inequality (4) holds, in fact if it does not hold, we have

\[
\sup_{x \in \Psi^{-1}(\rho, \infty]} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} > \beta,
\]

that implies \(b(\rho) < a(\rho)\) against the hypothesis.

Since \(\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = a(\rho)\), there exists \(\lambda_\rho \in [\alpha, \beta]\) such that

\[
\inf_{x \in X} (\Phi(x) + \lambda_\rho(\Psi(x) - \rho)) = a(\rho),
\]

thus we have

\[
\sup_{x \in \Psi^{-1}(\rho, \infty]} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} \leq \lambda_\rho \leq \inf_{x \in \Psi^{-1}(-\infty, \rho]} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)},
\]

that implies (5).

\((iiI) \Rightarrow (iI)\)

Let \(\rho \in \inf_{X} \Psi, \sup_{X} \Psi\) be such that (2) and (3) hold. In the previous proof "\((iiI) \Rightarrow (iiI)\)" we saw that (2) \(\iff b(\rho) < a(\rho)\), then at first we have

\[
\inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)) = b(\rho).
\]

Put

\[
A_\rho = \left[ \sup_{x \in \Psi^{-1}(\rho, \infty]} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)}, \inf_{x \in \Psi^{-1}(-\infty, \rho]} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)} \right] \cap [\alpha, \beta],
\]
it is $A_\rho \neq \emptyset$. In fact, $A_\rho$ is empty iff

\[ \inf_{x \in \Psi^{-1}(-\infty, \rho]} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)} < \alpha \text{ or } \sup_{x \in \Psi^{-1}(\rho, +\infty]} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)} > \beta. \]

If it is

\[ \inf_{x \in \Psi^{-1}(-\infty, \rho]} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)} < \alpha, \]

then $a(\rho) < b(\rho)$ that contradicts (2).

If it is

\[ \sup_{x \in \Psi^{-1}(\rho, +\infty]} \frac{\Phi(x) - b(\rho)}{\rho - \Psi(x)} > \beta, \]

the absurd

\[ \Phi(\tilde{x}) + \beta(\Psi(\tilde{x}) - \rho) < \inf_{x \in \Psi^{-1}(\rho, +\infty]} (\Phi(x) + \beta(\Psi(x) - \rho)), \]

for some $\tilde{x} \in \Psi^{-1}(\rho, +\infty]$, is obtained.

Thus we can choose $\lambda_\rho \in A_\rho$ for which $\inf_{x \in X} \Phi(x) + \lambda_\rho(\Psi(x) - \rho) \geq b(\rho)$, that implies the equality

\[ \sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = b(\rho) = \inf_{\lambda \in [\alpha, \beta]} \sup_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)). \]

Let $\rho \in ]\inf X, \sup X [\psi, \sup X [\psi$ be such that (4) and (5) hold. It is

\[ \inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)) = a(\rho). \]

Put

\[ B_\rho = \left[ \sup_{x \in \Psi^{-1}(\rho, +\infty]} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)}, \inf_{x \in \Psi^{-1}(-\infty, \rho]} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} \right] \cap [\alpha, \beta], \]
it is $B_\rho \neq \emptyset$. In fact, $B_\rho$ is empty iff

$$\inf_{x \in \Psi^{-1}([-\infty, \rho])} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} < \alpha \quad \text{or} \quad \sup_{x \in \Psi^{-1}([\rho, +\infty])} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} > \beta.$$  

The second inequality contradicts (4).

The first inequality implies that, for some $\bar{x} \in \Psi^{-1}([-\infty, \rho])$,

$$\Phi(\bar{x}) + \alpha(\Psi(\bar{x}) - \rho) < \inf_{x \in \Psi^{-1}([-\infty, \rho])} (\Phi(x) + \alpha(\Psi(x) - \rho)),$$

and this is absurd.

Thus we can choose $\lambda_\rho \in B_\rho$, then $\inf_{x \in X} (\Phi(x) + \lambda_\rho(\Psi(x) - \rho)) \geq a(\rho)$, that implies the equality

$$\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = a(\rho) = \inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)).$$

It is easily seen that, if $\rho \in \mathbb{R} \setminus \inf_x \Psi$, $\sup_x \Psi \neq \emptyset$, the equality

$$\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda(\Psi(x) - \rho))$$

holds.

**Corollary 1** Fix arbitrarily $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$, then for any $\rho \in \mathbb{R}$, one has

$$\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda(\Psi(x) - \rho)).$$

**Proof** Let us consider arbitrary $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$, by Lemma 1 it is enough to prove that (iii1) is true.

Let $\rho \in \mathbb{R} \setminus \inf_x \Psi$, $\sup_x \Psi \neq \emptyset$, since $a(\rho) \leq \inf_{\Psi^{-1}([-\infty, \rho])} \Phi$ the following inequalities hold:

$$\sup_{x \in \Psi^{-1}([\rho, +\infty])} \frac{\Phi(x) - a(\rho)}{\rho - \Psi(x)} \leq \sup_{x \in \Psi^{-1}([-\infty, \rho])} \frac{\Phi(x) - \inf_{\Psi^{-1}([-\infty, \rho])} \Phi}{\rho - \Psi(x)}$$
and

\[ \inf_{x \in \Psi^{-1}(]-\infty, 0[)} \frac{\Phi(x) - \inf_{\Psi^{-1}(]-\infty, 0[)} \Phi}{\rho - \Psi(x)} \leq \inf_{x \in \Psi^{-1}(]-\infty, 0[)} \frac{\Phi(x) - a(p)}{\rho - \Psi(x)} , \]

owing to (ii), then we have

\[ \sup_{x \in \Psi^{-1}(]p, +\infty[)} \frac{\Phi(x) - a(p)}{\rho - \Psi(x)} \leq \inf_{x \in \Psi^{-1}(]-\infty, 0[)} \frac{\Phi(x) - a(p)}{\rho - \Psi(x)} . \]

So, if \( a(p) \leq b(p) \) then (4) and (5) hold.

If \( b(p) < a(p) \), then (2) holds, moreover since \( b(p) < \inf_{\Psi^{-1}(]-\infty, 0[)} \Phi \) and by (ii), we also have (3).

**Lemma 2** Let \( a, \beta \in \mathbb{R}^+ \) with \( a < \beta \) and \( \rho \in \inf_{\Psi}, \sup_{\Psi}[ \) such that \( b(p) < a(p) \), then \( \Psi^{-1}(\rho) \neq \emptyset \) and for any \( \gamma \in [\alpha, \beta] \) one has

\[ \inf_{x \in \Psi^{-1}(]-\infty, 0[)} (\Phi(x) + \gamma(\Psi(x) - \rho)) = \inf_{x \in \Psi^{-1}(\rho)} \Phi(x). \]

**Proof** Fix \( \alpha, \beta \in \mathbb{R}^+ \) with \( \alpha < \beta \) and \( \rho \in \inf_{\Psi}, \sup_{\Psi}[ \) such that \( b(p) < a(p) \), by (ii), (2) of (ii) Lemma 1 and \( a(p) \leq \inf_{\Psi^{-1}(]-\infty, 0[)} \Phi \), we have

\[ \inf_{x \in \Psi^{-1}(]-\infty, 0[)} \frac{\Phi(x) - \inf_{\Psi^{-1}(]-\infty, 0[)} \Phi}{\rho - \Psi(x)} > \beta. \]

Then, for every \( \gamma \in [0, \beta] \), since \( \inf_{x \in \Psi^{-1}(]-\infty, 0[)} (\Phi(x) + \gamma(\Psi(x) - \rho)) \leq \inf_{\Psi^{-1}(]-\infty, 0[)} \Phi \), it turns out that

\[ \inf_{x \in \Psi^{-1}(]-\infty, 0[)} \frac{\Phi(x) - c(p)}{\rho - \Psi(x)} > \gamma, \]

where \( c(p) = \inf_{x \in \Psi^{-1}(]-\infty, 0[)} (\Phi(x) + \gamma(\Psi(x) - \rho)) \). Thus for every \( x \in \Psi^{-1}(]-\infty, \rho[) \),

\[ \Phi(x) + \gamma(\Psi(x) - \rho) > \inf_{x \in \Psi^{-1}(]-\infty, 0[)} (\Phi(x) + \gamma(\Psi(x) - \rho)), \]
moreover there exists \( x^* \in \Psi^{-1}([-\infty, \rho]) \) such that \( \Phi(x^*) + \gamma(\Psi(x^*) - \rho) = c(\rho) \), so it is necessarily \( x^* \in \Psi^{-1}(\rho) \) and
\[
\inf_{x \in \Psi^{-1}([-\infty, \rho])} (\Phi(x) + \gamma(\Psi(x) - \rho)) = \inf_{x \in \Psi^{-1}(\rho)} \Phi(x).
\]

**Lemma 3** Consider \( \alpha, \beta \in \mathbb{R}^+ \) with \( \alpha < \beta \) and a subdivision \( \alpha = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-2} \leq \alpha_{n-1} = \beta \) of the interval with \( n \geq 3 \). Define the function
\[
h(\lambda) = \sum_{i=1}^{n-1} \chi_{[\alpha_i, \alpha_{i+1}]}(\lambda)(\rho_i \lambda + a_i) \quad \text{for each } \lambda \in [\alpha, \beta],
\]
where \( \{\rho_k\}_{k=1}^{n-1} \) is a non-increasing finite sequence of real numbers and \( a_{i+1} = a_i + (\rho_i - \rho_{i+1})a_{i+1} \) for \( 1 \leq i \leq n - 2 \), with \( a_1 \in \mathbb{R} \) arbitrarily chosen. Then one has
\[
\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \inf_{\lambda \in [\alpha, \beta]} \sup_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)).
\]
(6)

**Proof** By Corollary 1, we have
\[
\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \max_{1 \leq i \leq n-1} \inf_{x \in X} \sup_{\lambda \in [\alpha_i, \alpha_{i+1}]} (\Phi(x) + \lambda \Psi(x) + \rho_i + a_i),
\]
thus there exists \( j \in \{1, 2, \ldots, n - 1\} \) such that
\[
\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda))) = \inf_{x \in X} \sup_{\lambda \in [\alpha_j, \alpha_{j+1}]} (\Phi(x) + \lambda \Psi(x) + \rho_j + a_j).
\]
(7)

For the sake of simplicity, we denote
\[
f_i(x, \lambda) = \Phi(x) + \lambda \Psi(x) + \rho_i + a_i
\]
for \( 1 \leq i \leq n - 1 \) being \( x \) and \( \lambda \) on the respective domains.
**1st Step**

At first, we prove the thesis when $\inf_x \Psi < -\rho_i < \sup_x \Psi$ for every $1 \leq i \leq n - 1$. Fix $1 \leq i \leq n - 2$ for every $x \in \Psi^{-1}([-\infty, -\rho_{i+1}])$ and $i \leq k \leq n - 1$, we have

\[
\sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} \left( \Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k \right) \geq \sup_{\lambda \in [\alpha_{k+1}, \alpha_{k+2}]} \left( \Phi(x) + \lambda(\Psi(x) + \rho_{k+1}) + a_{k+1} \right),
\]

hence

\[
\max_{i \leq k \leq n - 1} \sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} \left( \Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k \right) = \sup_{\lambda \in [\alpha_{i+1}, \alpha_{i+2}]} \left( \Phi(x) + \lambda(\Psi(x) + \rho_i) + a_i \right). \tag{8}
\]

Fix $2 \leq i \leq n - 2$, if $\Psi^{-1}([-\rho_i, -\rho_{i+1}]) \neq \emptyset$, then for every $x \in \Psi^{-1}([-\rho_i, -\rho_{i+1}])$ and $2 \leq k \leq i$, we have

\[
\sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} \left( \Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k \right) \geq \sup_{\lambda \in [\alpha_{k-1}, \alpha_k]} \left( \Phi(x) + \lambda(\Psi(x) + \rho_{k-1}) + a_{k-1} \right),
\]

whence, by (8), it follows that

\[
\max_{1 \leq k \leq n - 1} \sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} \left( \Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k \right) = \Phi(x) + \alpha_{i+1}(\Psi(x) + \rho_i) + a_i.
\]

For every $x \in \Psi^{-1}([-\rho_{n-1}, +\infty])$ and $1 \leq k \leq n - 2$, we have

\[
\sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} \left( \Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k \right) \geq \sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} \left( \Phi(x) + \lambda(\Psi(x) + \rho_{k+1}) + a_{k+1} \right),
\]
hence

\[
\max_{1 \leq k \leq n-1} \sup_{\lambda \in [\alpha_k, \alpha_{k+1}]} (\Phi(x) + \lambda(\Psi(x) + \rho_k) + a_k) = \Phi(x) + \beta(\Psi(x) + \rho_{n-1}) + a_{n-1}.
\]

We set \( N = \{1 \leq i \leq n - 2/\Psi^{-1}([-\rho_i, -\rho_{i+1}]) \neq \emptyset\} \) and

\[
\delta = \begin{cases} 
+\infty & \text{if } N = \emptyset \\
\min_{i \in N} \inf_{x \in \Psi^{-1}([-\rho_i, -\rho_{i+1}])} f_i(x, \alpha_{i+1}) & \text{if } N \neq \emptyset.
\end{cases}
\]

then it follows that

\[
\inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) = \min\left\{\delta, \inf_{x \in \Psi^{-1}([-\infty, -\rho_i])} f_1(x, \alpha), \inf_{x \in \Psi^{-1}([-\rho_i, +\infty])} f_{n-1}(x, \beta)\right\}. \tag{9}
\]

Now we state and prove the following assertions:

(a) If

\[
\inf_{x \in \Psi^{-1}([-\rho_j, +\infty])} f_j(x, \alpha_{j+1}) < \min\left\{\delta, \inf_{x \in \Psi^{-1}([-\infty, -\rho_i])} f_j(x, \alpha_j)\right\},
\]

then for every \( j \leq k \leq n - 1 \), we have

\[
\inf_{x \in \Psi^{-1}([-\rho_k, +\infty])} f_k(x, \alpha_{k+1}) \leq \inf_{x \in \Psi^{-1}([-\rho_i, +\infty])} f_j(x, \alpha_{j+1}) < \inf_{x \in \Psi^{-1}([-\infty, -\rho_k])} f_k(x, \alpha_k).
\]

(b) If

\[
\inf_{x \in \Psi^{-1}([-\infty, -\rho_j])} f_j(x, \alpha_j) \leq \inf_{x \in \Psi^{-1}([-\rho_i, +\infty])} f_j(x, \alpha_{j+1})
\]

and

\[
\inf_{x \in \Psi^{-1}([-\infty, -\rho_j])} f_j(x, \alpha_j) < \delta,
\]
then for every $1 \leq k \leq j$, we have

$$\inf_{x \in \Psi^{-1}([-\infty,-\rho])} f_k(x, \alpha_k) = \inf_{x \in \Psi^{-1}([-\infty,-\rho])} f_j(x, \alpha_j).$$

Let us prove (a).

If $j = n - 1$ the thesis is obvious.

Let $j < n - 1$, inequalities obviously hold for $k = j$. Put $T = \{j \leq k \leq n - 1\}$ such that the inequalities hold, then $T \neq \emptyset$ since $j \in T$. Let $m \in T$ with $m \leq n - 2$, we denote $A_{m,m+1} = \Psi^{-1}([-\rho_m,-\rho_{m+1}])$, then

$$\inf_{x \in \Psi^{-1}([-\infty,-\rho_{m+1}])} f_{m+1}(x, \alpha_{m+1})$$

$$= \inf_{x \in \Psi^{-1}([-\infty,-\rho_{m+1}])} f_m(x, \alpha_{m+1})$$

$$= \min \left\{ \begin{array}{ll}
\inf_{x \in \Psi^{-1}([-\infty,-\rho_m])} f_m(x, \alpha_{m+1}) & \text{if } A_{m,m+1} = \emptyset, \\
\inf_{x \in A_{m,m+1}} f_m(x, \alpha_{m+1}) & \text{if } A_{m,m+1} \neq \emptyset.
\end{array} \right. \tag{10}$$

Since $m \in T$,

$$\inf_{x \in \Psi^{-1}([-\rho_m,\infty])} (\Phi(x) + \alpha_{m+1}(\Psi(x) + \rho_m))$$

$$< \inf_{x \in \Psi^{-1}([-\infty,-\rho_m])} (\Phi(x) + \alpha_m(\Psi(x) + \rho_m)),$$

then, by Lemma 2, we have $\Psi^{-1}(-\rho_m) \neq \emptyset$ and

$$\inf_{x \in \Psi^{-1}([-\infty,-\rho_m])} (\Phi(x) + \alpha_{m+1}(\Psi(x) + \rho_m))$$

$$= \inf_{x \in \Psi^{-1}([-\infty,-\rho_m])} (\Phi(x) + \alpha_m(\Psi(x) + \rho_m)).$$
so one has
\[ \inf_{x \in \Psi^{-1}(]-\rho_j, +\infty[)} f_j(x, \alpha_{j+1}) < \inf_{x \in \Psi^{-1}(]-\infty, -\rho_m[)} f_m(x, \alpha_{m+1}). \]

Moreover, if \( A_{m, m+1} \neq \emptyset \), then the hypotheses imply that
\[ \inf_{x \in \Psi^{-1}(]-\rho_j, +\infty[)} f_j(x, \alpha_{j+1}) < \inf_{x \in \Psi^{-1}(]-\rho_m, -\rho_{m+1}[)} f_m(x, \alpha_{m+1}). \]

Consequently, by (10), it follows
\[ \inf_{x \in \Psi^{-1}(]-\rho_j, +\infty[)} f_j(x, \alpha_{j+1}) < \inf_{x \in \Psi^{-1}(]-\rho_m, -\rho_{m+1}[)} f_{m+1}(x, \alpha_{m+1}). \quad (11) \]

By (7), we have
\[ \inf_{x \in \Psi^{-1}(]-\rho_j, +\infty[)} f_j(x, \alpha_{j+1}) = \max_{1 \leq i \leq n-1} \inf_{x \in X} \sup_{\lambda \in [\alpha_i, \alpha_{i+1}]} f_i(x, \lambda), \]

hence
\[ \inf_{x \in \Psi^{-1}(]-\rho_j, +\infty[)} f_j(x, \alpha_{j+1}) \geq \inf_{x \in X} \sup_{\lambda \in [\alpha_m, \alpha_{m+2}]} f_{m+1}(x, \lambda) \]
\[ = \min \left\{ \inf_{x \in \Psi^{-1}(]-\rho_{m+1}[)} f_{m+1}(x, \alpha_{m+1}), \inf_{x \in \Psi^{-1}(]-\rho_{m+1}, +\infty[)} f_{m+1}(x, \alpha_{m+2}) \right\}, \]

therefore, by (11), it turns out that
\[ \inf_{x \in \Psi^{-1}(]-\rho_j, +\infty[)} f_j(x, \alpha_{j+1}) \geq \inf_{x \in \Psi^{-1}(]-\rho_{m+1}, +\infty[)} f_{m+1}(x, \alpha_{m+2}). \]

Thus we have proved that \( m \in T \) with \( m \leq n - 2 \Rightarrow m + 1 \in T \), then the thesis of assertion (a) is proved.

Let us prove (b).

If \( j = 1 \) the thesis is obvious.

Let \( j > 1 \), the equality obviously holds when \( k = j \). Put \( T = \{ 1 \leq k \leq j \mid \text{the equality holds} \} \), then \( T \neq \emptyset \) because \( j \in T \). Suppose \( m \in T \) with \( m > 1 \),
we have that

\[
\inf_{x \in \Psi^{-1}([-\infty, -\rho_m])} f_m(x, \alpha_m) = \inf_{x \in \Psi^{-1}([-\infty, -\rho_m])} f_{m-1}(x, \alpha_m)
\]

\[
= \begin{cases} 
\inf_{x \in \Psi^{-1}([-\infty, -\rho_{m-1}])} f_{m-1}(x, \alpha_m) & \text{if } \mathcal{A}_{m-1,m} = \emptyset, \\
\min \left\{ \inf_{x \in \Psi^{-1}([-\infty, -\rho_{m-1}])} f_{m-1}(x, \alpha_m), \inf_{x \in \mathcal{A}_{m-1,m}} f_{m-1}(x, \alpha_m) \right\} & \text{if } \mathcal{A}_{m-1,m} \neq \emptyset,
\end{cases}
\]

(12)

then it is seen, by similar arguments to those in the proof of assertion (a), that

\[
\inf_{x \in \Psi^{-1}([-\infty, -\rho_m])} f_m(x, \alpha_m) = \inf_{x \in \Psi^{-1}([-\infty, -\rho_{m-1}])} f_{m-1}(x, \alpha_m).
\]

Because of (7) and \( m \in T \), we also have

\[
\inf_{x \in \Psi^{-1}([-\infty, -\rho_{m-1}])} f_{m-1}(x, \alpha_{m-1}) = \inf_{x \in \Psi^{-1}([-\infty, -\rho_{j}])} f_{j}(x, \alpha_j)
\]

\[
\geq \min \left\{ \inf_{x \in \Psi^{-1}([-\infty, -\rho_{m-1}])} f_{m-1}(x, \alpha_{m-1}), \inf_{x \in \Psi^{-1}([-\rho_{m-1}, +\infty])} f_{m-1}(x, \alpha_{m}) \right\},
\]

which implies, by Lemma 2,

\[
\inf_{x \in \Psi^{-1}([-\infty, -\rho_{m-1}])} f_{m-1}(x, \alpha_{m}) \geq \inf_{x \in \Psi^{-1}([-\infty, -\rho_{m-1}])} f_{m-1}(x, \alpha_{m-1}).
\]

Since \( \alpha_{m-1} \leq \alpha_m \), the opposite inequality holds too, therefore the thesis is proved.

Now we can prove the equality (6) with the further hypothesis we stated at the beginning of this step.

If

\[
\inf_{x \in \Psi^{-1}([-\rho_{j}, +\infty])} f_{j}(x, \alpha_{j+1}) < \inf_{x \in \Psi^{-1}([-\infty, -\rho_{j}])} f_{j}(x, \alpha_{j}),
\]
then by (7),
\[
\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \inf_{x \in \Psi^{-1}([-\rho, +\infty[)} f_j(x, \alpha_{j+1}),
\]
owing to (a), it follows that
\[
\inf_{x \in \Psi^{-1}([-\rho, +\infty[)} f_j(x, \alpha_{j+1}) \geq \delta
\]
or
\[
\inf_{x \in \Psi^{-1}([-\rho, +\infty[)} f_j(x, \alpha_{j+1}) \geq \inf_{x \in \Psi^{-1}([-\rho_{n-1}, +\infty[)} f_{n-1}(x, \beta),
\]
hence
\[
\sup \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) \geq \inf \sup_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)),
\]
that implies (6).

If
\[
\inf_{x \in \Psi^{-1}([-\rho, +\infty[)} f_j(x, \alpha_j) \leq \inf_{x \in \Psi^{-1}([-\rho, +\infty[)} f_j(x, \alpha_{j+1}),
\]
then the equality (6) is implied by assertion (b).

2nd Step

We prove that the equality (6) holds, when \(\inf_{x \in X} \Psi(x) \neq -\infty\) and there exists \(1 \leq \tilde{k} \leq n - 2\) such that

for \(1 \leq i \leq \tilde{k}\)
\[-\rho_i \leq \inf_{x \in X} \Psi(x),\]

for \(\tilde{k} + 1 \leq i \leq n - 1\)
\[\inf_{x \in X} \Psi(x) < -\rho_i < \sup_{x \in X} \Psi(x).\]

We have
\[
\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \max_{k+1 \leq i \leq n-1} \inf_{x \in X} \sup_{\lambda \in [\alpha_i, \alpha_{i+1}]} (\Phi(x) + \lambda (\Psi(x) + \rho_i) + a_i), \tag{13}
\]
and
\[
\inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \inf_{x \in X} \max_{k+1 \leq i \leq n-1} \sup_{\lambda \in [\alpha_i, \alpha_{i+1}]} (\Phi(x) + \lambda(\Psi(x) + \rho_i) + a_i). \tag{14}
\]

Put \( g = h|_{[\alpha_{k+1}, \beta]} \), then the equality (6) follows from the 1st step, where \( g \) takes the place of \( h \), and from (13), (14).

3rd Step
Let for \( 1 \leq i \leq n - 1 \), \( -\rho_i \leq \inf_{X} \Psi \neq -\infty \), then we have
\[
\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \inf_{x \in X} f_{n-1}(x, \beta) = \inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda \Psi(x) + h(\lambda)).
\]

Let for \( 1 \leq i \leq n - 1 \), \( -\rho_i \geq \sup_{X} \Psi(x) \neq +\infty \), then we have
\[
\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \inf_{x \in X} f_{1}(x, \alpha) = \inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda \Psi(x) + h(\lambda)).
\]

4th Step
Suppose that there exists \( 2 \leq \bar{k} \leq n - 1 \) such that

for \( 1 \leq i \leq \bar{k} - 1 \), \( -\rho_i \leq \sup_{x \in X} \Psi(x) \),

for \( \bar{k} \leq i \leq n - 1 \), \( -\rho_i \geq \sup_{x \in X} \Psi(x) \).

We have
\[
\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \sup_{\lambda \in [\alpha, \alpha_{\bar{k}}]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + g(\lambda))
\]

and
\[
\inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) = \inf_{x \in X} \sup_{\lambda \in [\alpha, \alpha_{\bar{k}}]} (\Phi(x) + \lambda \Psi(x) + g(\lambda)),
\]

where \( g = h|_{[\alpha, \alpha_{\bar{k}}]} \). Therefore the equality (6) follows from the previous steps, where \( g \) takes the place of \( h \).
Lemma 4 Let $\alpha, \beta \in \mathbb{R}^+$ with $\alpha < \beta$ and $g : [\alpha, \beta] \to \mathbb{R}$ be a concave function such that $\max \{|g'_d(\alpha)|, |g'_s(\beta)|\} \neq +\infty$. There exists a non-increasing sequence of functions $\{g_n\}_{n \in \mathbb{N}}$ pointwise convergent to $g$ on $[\alpha, \beta]$ such that for every $n \in \mathbb{N}$, $g_n$ is formally defined as the function $h$ in Lemma 3.

Proof Fix $n \in \mathbb{N}$, we set

$$\delta_k^{(n)} = \alpha + k \frac{\beta - \alpha}{2^{n-1}} \quad \text{for } 0 \leq k \leq 2^{n-1};$$

$$\rho_0^{(n)} = g'_d(\alpha), \quad \rho_{2^n}^{(n)} = g'_s(\beta) \quad \text{and for } 1 \leq j \leq 2^n - 1$$

$$\rho_j^{(n)} = \begin{cases} 
    g'_s(\delta_k^{(n)}) & \text{with } k = \frac{j + 1}{2}, \quad \text{if } j \text{ is odd}, \\
    g'_d(\delta_k^{(n)}) & \text{with } k = \frac{j}{2}, \quad \text{if } j \text{ is even},
  \end{cases}$$

$$\alpha_j^{(n)} = g(\delta_k^{(n)}) - \rho_j^{(n)} \delta_k^{(n)} \quad \text{for } 0 \leq j \leq 2^n,$$

$$\alpha_0^{(n)} = \alpha, \quad \alpha_{2^n+1}^{(n)} = \beta \quad \text{and for } 1 \leq j \leq 2^n$$

$$\alpha_j^{(n)} = \begin{cases} 
    \delta_k^{(n)} & \text{if } \rho_j^{(n)} = \rho_{j-1}^{(n)}, \\
    \frac{\alpha_j^{(n)} - \alpha_{j+1}^{(n)}}{\rho_j^{(n)} - \rho_{j+1}^{(n)}} & \text{if } \rho_j^{(n)} \neq \rho_{j+1}^{(n)},
  \end{cases}$$

with $k = j/2$ if $j$ is even, $k = (j + 1)/2$ if $j$ is odd.

Because $g$ is concave, then $\{\rho_k^{(n)}\}_{0 \leq k \leq 2^n}$ is non-increasing, moreover it is easily seen that $\{\alpha_k^{(n)}\}_{0 \leq k \leq 2^n+1}$ is a subdivision of the interval $[\alpha, \beta]$.

Now we define the function:

$$g_n(\lambda) = \sum_{j=0}^{2^n} \chi_{[\alpha_j^{(n)}, \alpha_{j+1}^{(n)}]}(\lambda) (\rho_j^{(n)} \lambda + a_j^{(n)}) \quad \text{for each } \lambda \in [\alpha, \beta],$$

from the definition of $\alpha_{j+1}^{(n)}$, one has $a_{j+1}^{(n)} = a_j^{(n)} + (\rho_j^{(n)} - \rho_{j+1}^{(n)}) \alpha_{j+1}^{(n)}$ for every $0 \leq j \leq 2^n-1$. 
We prove that \( \{g_n\} \) is pointwise convergent to \( g \) on the interval \([\alpha, \beta]\): Fix \( \lambda \in [\alpha, \beta] \) and \( n \in \mathbb{N} \), we put \( k_n = \max\{0 \leq k \leq 2^{n-1}: \delta_k^{(n)} \leq \lambda\} \); since \( \alpha_{2k_n}^{(n)} = \delta_k^{(n)} \) and \( \alpha_{2k_n+2}^{(n)} = \delta_{k_n+1}^{(n)} \), one has \( \lambda \in [\alpha_{2k_n}^{(n)}, \alpha_{2k_n+2}^{(n)}] \), then

\[
|g_n(\lambda) - g(\lambda)| \leq \max\{|g'_d(\alpha)|, |g'_s(\beta)|\} \frac{\beta - \alpha}{2^{n-1}}
+ \max\{|g(\delta_{k_n}^{(n)}) - g(\lambda)|, |g(\delta_{k_n+1}^{(n)}) - g(\lambda)|\}.
\]

The function \( g \) is continuous, then, since

\[
\lim_{n \to \infty} \delta_{k_n}^{(n)} = \lim_{n \to \infty} \delta_{k_n+1}^{(n)} = \lambda,
\]

we have

\[
\lim_{n \to \infty} |g_n(\lambda) - g(\lambda)| = 0.
\]

Let \( n \in \mathbb{N} \) and \( \lambda \in [\alpha, \beta] \), it results that \( g_n(\lambda) \geq g_{n+1}(\lambda) \).

There exists \( 0 \leq j \leq 2^{n+1} \) such that \( \lambda \in [\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}] \), then we have to examine the different cases which can occur.

If \( j \) is even and \( k = j/2 \) is also even, for some \( 0 \leq m \leq 2^{n-1} \), one has \( j = 2k = 4m \). In this case, at first, we have

\[
[\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}] \subseteq [\alpha_k^{(n)}, \alpha_{k+1}^{(n)}],
\]

hence

\[
g_n(\lambda) - g_{n+1}(\lambda) = (\rho_k^{(n)} \lambda + a_k^{(n)}) - (\rho_j^{(n+1)} \lambda + a_j^{(n+1)}) = 0.
\]

If \( j \) is even and \( k = j/2 \) is odd, then \( j = 2k = 4m + 2 \) for some \( 0 \leq m \leq 2^{n-1} - 1 \). In this case, it results that

\[
[\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}] \subseteq [\alpha_k^{(n)}, \alpha_{k+1}^{(n)}].
\]

Since for every \( \lambda \in ]\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}[ \cap ]\alpha_k^{(n)}, \alpha_{k+1}^{(n)}[ \), we have

\[
g_n'(\lambda) - g_{n+1}'(\lambda) = \rho_{2m}^{(n)} - \rho_{4m+2}^{(n)} \geq 0
\]
and

\[ g_n(\alpha_j^{(n+1)}) - g_{n+1}(\alpha_j^{(n+1)}) = (\rho_m^{(n)} - \rho_{m+1}^{(n+1)}) (\alpha_m^{(n+1)} - \alpha_{m+1}^{(n+1)}) \geq 0, \]

it follows that

\[ g_n(\lambda) \geq g_{n+1}(\lambda), \quad \text{if } \lambda \in [\alpha_m^{(n+1)}, \alpha_{m+1}^{(n+1)}] \cap [\alpha_k^{(n)}, \alpha_{k+1}^{(n)}]. \]

Since for every \( \lambda \in [\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}] \cap [\alpha_k^{(n)}, \alpha_{k+1}^{(n)}] \), one has

\[ g'_n(\lambda) - g'_{n+1}(\lambda) \leq 0, \]

and

\[ g_n(\alpha_j^{(n+1)}) - g_{n+1}(\alpha_{j+1}^{(n+1)}) = 0, \]

it follows that

\[ g_n(\lambda) \geq g_{n+1}(\lambda), \quad \text{if } \lambda \in [\alpha_j^{(n+1)}, \alpha_{j+1}^{(n+1)}] \cap [\alpha_k^{(n)}, \alpha_{k+1}^{(n)}]. \]

In the remaining cases, the inequality \( g_n(\lambda) \geq g_{n+1}(\lambda) \) also holds, the proof is analogous to the previous ones.

3. PROOF OF THEOREM 1

In the first instance, we remark that if \( \rho \in ]\inf \Psi, \sup \Psi[ \), then

\[ \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{\Psi^{-1}(-\infty,0]} \Phi. \]

\((i) \Rightarrow (ii)\)

Let \( \rho \in ]\inf \Psi, \sup \Psi[ \), then \( \lim_{\lambda \to +\infty} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = -\infty \), moreover the function \( \lambda \in [0, +\infty[ \to \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) \) is u.s.c. then attains its supremum. Consequently there exists \( \lambda_1 \in [0, +\infty[ \)
such that
\[
\inf_{x \in X} (\Phi(x) + \lambda_\rho(\Psi(x) - \rho)) = \sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{\Psi^{-1}([-\infty, \rho[)} \Phi,
\]
so we have
\[
\sup_{x \in \Psi^{-1}([\rho, +\infty[)} \frac{\Phi(x) - \inf_{\Psi^{-1}([-\infty, \rho[)} \Phi}{\rho - \Psi(x)} \leq \lambda_\rho \leq \inf_{x \in \Psi^{-1}([-\infty, \rho[)} \frac{\Phi(x) - \inf_{\Psi^{-1}([-\infty, \rho[)} \Phi}{\rho - \Psi(x)},
\]
then owing to arbitrariness of \(\rho\) the thesis is proved.

\((ii) \Rightarrow (i)\)

Let \(\rho \in [\inf_X \Psi, \sup_X \Psi[, \) since
\[
0 \leq \inf_{x \in \Psi^{-1}([-\infty, \rho[)} \frac{\Phi(x) - \inf_{\Psi^{-1}([-\infty, \rho[)} \Phi}{\rho - \Psi(x)} < +\infty,
\]
we can set
\[
\lambda_\rho = \inf_{x \in \Psi^{-1}([-\infty, \rho[)} \frac{\Phi(x) - \inf_{\Psi^{-1}([-\infty, \rho[)} \Phi}{\rho - \Psi(x)},
\]
so we have
\[
\inf_{x \in X} (\Phi(x) + \lambda_\rho(\Psi(x) - \rho)) \geq \inf_{\Psi^{-1}([-\infty, \rho[)} \Phi = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\Psi(x) - \rho)),
\]
therefore the equality
\[
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\Psi(x) - \rho))
\]
holds.
In order to complete the proof we have to prove that, if $R \setminus \inf_X \Psi, \sup_X \Psi \neq \emptyset$, for every $\rho \in R \setminus \inf_X \Psi, \sup_X \Psi \}$ the equality in (i) holds.

If $\sup_X \Psi \neq +\infty$ and $\rho \geq \sup_X \Psi$ then

$$\inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{x \in X} \Phi(x),$$

thus the equality follows because

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) \geq \inf_{x \in X} \Phi(x).$$

Now we suppose that $\inf_X \Psi \neq -\infty$ and $\rho \leq \inf_X \Psi$. It is necessary to distinguish the following two cases:

(1) $\Psi$ does not have absolute minimum.
(2) $\Psi$ has absolute minimum.

Let (1) be true.

Since for every $x \in X$, $\Psi(x) - \rho > 0$, it follows that $\inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\Psi(x) - \rho)) = +\infty$. We assume that

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\Psi(x) - \rho)),$$

then there exists $\alpha \in R$ such that $\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \alpha$. Consequently, for every $n \in N$, there exists $x_n \in X$ such that $\Phi(x_n) + n(\Psi(x_n) - \rho) < \alpha + 1$, since for every $n \in N$, $\Psi(x_n) - \rho > 0$, it follows $\Phi(x_n) < \alpha + 1$, then the weak coerciveness of $\Phi$ implies that $\{x_n\}_{n \in N}$ is bounded. Because of the hypotheses about $E$ and $X$, there exist $x^* \in X$ and a subsequence $\{x_{n_k}\}_{k \in N}$ such that $x_{n_k} \rightharpoonup x^*$ weakly for $k \to \infty$. The function $\Phi$ is weakly sequentially l.s.c., then

$$\Phi(x^*) + \liminf_{k \to \infty} n_k(\Psi(x_{n_k}) - \rho) \leq \liminf_{k \to \infty} \Phi(x_{n_k}) + n_k(\Psi(x_{n_k}) - \rho) \leq \alpha + 1,$$

consequently, it follows $\liminf_{k \to \infty} \Psi(x_{n_k}) = \rho$.
Therefore we have the absurd \( \rho < \Psi(x^\star) \leq \lim_{k \to \infty} \Psi(x_{n_k}) = \rho \), being \( \Psi \) weakly sequentially l.s.c. The absurd follows from the hypothesis that the equality in (i) does not hold, so the thesis is proved.

Let (2) be true.

If we choose \( \rho < \inf_{x} \Psi \), since for every \( x \in X \), \( \Psi(x) > \rho \), we can proceed as in (1).

Let \( \rho = \inf_{x} \Psi \), then

\[
\inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\Psi(x) - \rho)) = \inf_{x \in \Psi^{-1}(\rho)} \Phi(x),
\]

in fact, if we assume that \( \sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) < \inf_{x \in \Psi^{-1}(\rho)} \Phi(x) \), we can choose \( \gamma \in \mathbb{R} \) such that

\[
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - \rho)) < \gamma < \inf_{x \in \Psi^{-1}(\rho)} \Phi(x).
\]

Therefore for every \( \lambda \in [0, +\infty[ \) it results that

\[
\inf_{x \in X, \Psi(x) \neq \rho} (\Phi(x) + \lambda(\Psi(x) - \rho)) < \gamma,
\]

hence, for every \( n \in \mathbb{N} \), there exists \( x_n \in X \) with \( \Psi(x_n) > \rho \) and \( \Phi(x_n) + n(\Psi(x_n) - \rho) < \gamma \). Since for every \( n \in \mathbb{N} \), \( \Phi(x_n) < \gamma \), there exist \( x^\star \in X \) and a subsequence \( \{x_{n_k}\}_{k \in \mathbb{N}} \) such that \( x_{n_k} \to x^\star \) weakly when \( k \to \infty \), so \( \rho \leq \Psi(x^\star) \leq \lim_{k \to \infty} \Psi(x_{n_k}) = \rho \), that implies \( x^\star \in \Psi^{-1}(\rho) \). We also have

\[
\Phi(x^\star) \leq \lim_{k \to \infty} \Phi(x_{n_k}) \leq \gamma < \inf_{x \in \Psi^{-1}(\rho)} \Phi(x),
\]

that is absurd since \( x^\star \in \Psi^{-1}(\rho) \).

(iii) \( \Rightarrow \) (i) is obvious.

(ii) \( \Rightarrow \) (iii)

Consider a concave function \( h: [0, +\infty[ \to \mathbb{R} \), let \( 0 < \alpha < \beta \) be arbitrary real numbers and set \( g = h_{[\alpha, \beta]} \), the function \( g \) meets the hypotheses of Lemma 4, hence we can consider a non-increasing sequence of functions \( \{g_n\}_{n \in \mathbb{N}} \) pointwise convergent to \( g \) such that, for every \( n \in \mathbb{N} \),
by Lemma 3,
\[
\sup_{\lambda \in [\alpha, \beta]} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + g_n(\lambda)) = \inf_{x \in X} \sup_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda \Psi(x) + g_n(\lambda)).
\]
(15)

Since \([\alpha, \beta]\) is compact and \(\{g_n\}_{n \in \mathbb{N}}\) is a monotone sequence of functions pointwise convergent to \(g\), it follows that \(g_n \to g\) uniformly on \([\alpha, \beta]\) when \(n \to +\infty\), by the Dini’s theorem. Hence

\[
\sup_{x \in X} \inf_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda \Psi(x) + g_n(\lambda)) \to \sup_{x \in X} \inf_{\lambda \in [\alpha, \beta]} (\Phi(x) + \lambda \Psi(x) + g(\lambda)),
\]

where the last inequality is due to \(g(\lambda) = \inf_{n \in \mathbb{N}} g_n(\lambda)\) for any \(\lambda \in [\alpha, \beta]\). Thus the equality follows.

Since \(\beta > \alpha\) is arbitrary, it follows that

\[
\sup_{\lambda \in [\alpha, +\infty[} \inf_{\lambda \in [\alpha, \beta]} \sup_{x \in X} (\Phi(x) + \lambda \Psi(x) + g(\lambda))
\]

Let us suppose that

\[
\sup_{\beta > \alpha} \inf_{\lambda \in [\alpha, \beta]} \sup_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) < \inf_{\lambda \in [\alpha, +\infty[} \sup_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)),
\]

then we can choose \(\gamma \in \mathbb{R}\) such that

\[
\sup_{\beta > \alpha} \inf_{\lambda \in [\alpha, \beta]} \sup_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) < \gamma < \inf_{\lambda \in [\alpha, +\infty[} \sup_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)),
\]
consequently for every $\beta > \alpha$ there exists $x_\beta \in X$ such that
\[
\sup_{\lambda \in [\alpha, \beta]} (\Phi(x_\beta) + \lambda \Psi(x_\beta) + h(\lambda)) < \gamma
\]
in particular $\Phi(x_\beta) + \alpha \Psi(x_\beta) < \gamma - h(\alpha)$, that implies $\{x_\beta\}_{\beta > \alpha}$ is bounded owing to the weak coerciveness of the functional $\Phi(\cdot) + \alpha \Psi(\cdot)$. Therefore there exist $x^* \in X$ and a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ with $x_{n_k} \to x^*$ when $k \to +\infty$. Fix $\delta > \alpha$, there exists $\tilde{k} \in \mathbb{N}$ such that $n_k \geq \delta$ for each $k \geq \tilde{k}$, moreover the function $x \in X \to \sup_{\lambda \in [\alpha, \delta]} (\Phi(x) + \lambda \Psi(x) + h(\lambda))$ is weakly sequentially l.s.c., then it is
\[
\sup_{\lambda \in [\alpha, \delta]} (\Phi(x^*) + \lambda \Psi(x^*) + h(\lambda)) \leq \gamma.
\]
Because of the arbitrariness of $\delta > \alpha$, it follows that
\[
\sup_{\lambda \in [\alpha, +\infty[} (\Phi(x^*) + \lambda \Psi(x^*) + h(\lambda)) \leq \gamma,
\]
from which, we obtain the absurd
\[
\inf_{x \in X} \sup_{\lambda \in [\alpha, +\infty[} (\Phi(x) + \lambda \Psi(x) + h(\lambda))
\leq \gamma < \inf_{x \in X} \sup_{\lambda \in [\alpha, +\infty[} (\Phi(x) + \lambda \Psi(x) + h(\lambda)).
\]
Thus, at this point, we have for each $\alpha > 0$
\[
\sup_{\lambda \in [\alpha, +\infty[} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda))
= \inf_{x \in X} \sup_{\lambda \in [\alpha, +\infty[} (\Phi(x) + \lambda \Psi(x) + h(\lambda)),
\]
from which, it follows that
\[
\sup_{\lambda \in [0, +\infty[} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda))
= \sup_{\alpha > 0} \inf_{x \in X} \sup_{\lambda \in [\alpha, +\infty[} (\Phi(x) + \lambda \Psi(x) + h(\lambda)).
\]
It is easily seen, by similar arguments as above, that

\[
\sup_{\alpha > 0} \inf_{x \in X} \sup_{\lambda \in [\alpha, +\infty[} \left( \Phi(x) + \lambda \Psi(x) + h(\lambda) \right) = \inf_{x \in X} \sup_{\lambda \in [0, +\infty[} \left( \Phi(x) + \lambda \Psi(x) + h(\lambda) \right).
\]

Since \( h \) is concave, for each \( x \in X \) the function \( \lambda \in [0, +\infty[ \to \Phi(x) + \lambda \Psi(x) + h(\lambda) \) is l.s.c., then

\[
\sup_{\lambda \in [0, +\infty[} \left( \Phi(x) + \lambda \Psi(x) + h(\lambda) \right) = \sup_{\lambda \in [0, +\infty[} \left( \Phi(x) + \lambda \Psi(x) + h(\lambda) \right),
\]

therefore

\[
\sup_{\lambda \in [0, +\infty[} \inf_{x \in X} \left( \Phi(x) + \lambda \Psi(x) + h(\lambda) \right) = \inf_{x \in X} \sup_{\lambda \in [0, +\infty[} \left( \Phi(x) + \lambda \Psi(x) + h(\lambda) \right),
\]

that implies the thesis.

References