Existence Theory for Nonlinear Volterra Integral and Differential Equations

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In this paper we prove the existence theorems for the integrodifferential equation

\[ y'(t) = f \left( t, y(t), \int_0^t k(t, s, y(s)) \, ds \right), \quad t \in I = [0, T], \]
\[ y(0) = y_0, \]

where in first part \( f, k, y \) are functions with values in a Banach space \( E \) and the integral is taken in the sense of Bochner. In second part \( f, k \) are weakly–weakly sequentially continuous functions and the integral is the Pettis integral. Additionally, the functions \( f \) and \( k \) satisfy some boundary conditions and conditions expressed in terms of measure of noncompactness or measure of weak noncompactness.

Keywords: Integral equations; Existence theorem; Pseudo-solutions; Measures of noncompactness

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1 INTRODUCTION

In this paper we establish some existence principles for integrodifferential operator equations and present existence result for integrodifferential and integral equations.

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The paper is divided into two main sections. In Section 1 we prove some existence theorems for the problem

\[ y'(t) = f\left(t, y(t), \int_0^t k(t, s, y(s)) \, ds\right), \]
\[ y(0) = y_0, \]

where \( I = [0, T] \), \( E \) is a Banach space with the norm \( \| \cdot \| \), \( f, k, y \) are functions with values in a Banach space \( E \) and the integral is the Bochner integral.

In Section 2 we prove some existence theorem for the problem (1), where \( f, k, y \) are functions with values in a Banach space \( E \), \( f, k \) are functions weakly-weakly sequentially continuous and the integral is the Pettis integral [1]. The results of this paper extend existence theorems from Krzyśka [12], Cichoń [6], Meehan and O'Regan [13], O'Regan [16,17], Cramer et al. [7].

In this paper we use the measure of noncompactness developed by Kuratowski [11], and the measure of weak noncompactnes developed by de Blasi [4].

Let \( A \) be a bounded nonvoid subset of \( E \). The Kuratowski measure of noncompactness \( \alpha(A) \) is defined by

\[ \alpha(A) = \inf\{\varepsilon > 0: \text{there exists } C \in \mathcal{K} \text{ such that } A \subset C + \varepsilon B_0\}, \]

where \( \mathcal{K} \) is the set of compact subsets of \( E \) and \( B_0 \) is the norm unit ball.

The de Blasi measure of weak noncompactness \( \beta(A) \) is defined by

\[ \beta(A) = \inf\{t > 0: \text{there exists } C \in \mathcal{K}^w \text{ such that } A \subset C + tB_0\}, \]

where \( \mathcal{K}^w \) is the set of weakly compact subsets of \( E \) and \( B_0 \) is the norm unit ball.

The properties of measure of noncompactness \( \alpha(A) \) are:

\( (1^0) \) if \( A \subset B \) then \( \alpha(A) \leq \alpha(B) \);

\( (2^0) \) \( \alpha(A) = \alpha(\overline{A}) \), where \( \overline{A} \) denotes the closure of \( A \);

\( (3^0) \) \( \alpha(A) = 0 \) if and only if \( A \) is relatively compact;

\( (4^0) \) \( \alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\} \);
\( (5^0) \quad \alpha(\lambda A) = |\lambda|\alpha(A) \quad (\lambda \in \mathbb{R}) \);
\( (6^0) \quad \alpha(A + B) \leq \alpha(A) + \alpha(B) \);
\( (7^0) \quad \alpha(\text{conv } A) = \alpha(A) \).

The properties of weak measure of noncompactness \( \beta \) are analogous to the properties of measure of noncompactness, see \([2-5,14]\). Moreover, we can construct many other measures with the above properties, by using a scheme from \([5]\). We now gather some well-known definitions and results from the literature, which we will use throughout this paper.

**Definition 1** A function \( f : I \times E \times E \to E \) is \( L^1 \)-Carathéodory, if the following conditions hold:

(i) the map \( t \mapsto f(t, x, y) \) is measurable for all \( (x, y) \in E^2 \);

(ii) the map \( (x, y) \mapsto f(t, x, y) \) is continuous for almost all \( t \in I \).

**Definition 2** A function \( k : I \times I \times B \to E \) is \( L^1 \)-Carathéodory, if the following conditions hold:

(i) the map \( (t, s) \mapsto f(t, s, y) \) is measurable for all \( y \in B \);

(ii) the map \( y \mapsto f(t, s, y) \) is continuous for almost all \( (t, s) \in I^2 \).

In the proof of the main theorem in Section 1 we will apply the following fixed point theorem.

**Theorem 1** \([15]\) Let \( D \) be a closed convex subset of \( E \), and let \( F \) be a continuous map from \( D \) into itself. If for some \( x \in D \) the implication

\[
\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \implies V \text{ is relatively compact},
\]

holds for every countable subset \( V \) of \( D \), then \( F \) has a fixed point.

In Section 2 we will apply the following theorem:

**Theorem 2** \([10]\) Let \( E \) be a metrizable locally convex topological vector space and let \( D \) be a closed convex subset of \( E \), and let \( F \) be a weakly sequentially continuous map of \( D \) into itself. If for some \( x \in D \) the implication

\[
\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \implies V \text{ is relatively weakly compact},
\]

holds for every subset \( V \) of \( D \), then \( F \) has a fixed point.
2 AN EXISTENCE RESULT FOR INTEGRODIFFERENTIAL EQUATIONS

Observe that the problem (1) is equivalent to the integral equation

\[ y(t) = y_0 + \int_0^t f \left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz, \quad \text{for } t \in I. \quad (1') \]

Assume that

1. a function \( a \in L^1[0, T], \)
2. \( B = \{ x : \| x \| \leq b, b = \| y_0 \| + \int_0^T a(t) \, dt \}, \)
3. \( k \) is a \( L^1 \)-Carathéodory function from \( I^2 \times B \) into \( E, \)
4. \( f \) is a \( L^1 \)-Carathéodory function from \( I \times B \times B \) into \( E, \)
5. \( \| f(t, y(t), \int_0^t k(t, s, y(s)) \, ds) \| \leq a(t) \) almost everywhere on \( I \) for \( y \in \bar{B}, \) where \( \bar{B} = \{ y \in C[0, T] : \| y \| \leq b, b = \| y_0 \| + \int_0^T a(t) \, dt \}. \)

**Theorem 3** Assume, that conditions (1)–(5) holds and in addition, that

6. there exists a constant \( c_1 \) such that \( \alpha(f(t, A, C)) \leq c_1 \max\{ \alpha(A), \alpha(C) \}, \) for any subsets \( A, C \) of \( B, \)
7. there exists an integrable function \( c_2 : I^2 \to \mathbb{R}^+ \) such that for every \( t \in I, \)
\( \varepsilon > 0 \) and for every bounded subset \( X \) of \( B \) there exists a closed subset \( I_\varepsilon \) of \( I \) such that \( \text{mes}(I \setminus I_\varepsilon) < \varepsilon \) and
\[ \alpha(k(t, T \times X)) \leq \sup_{s \in T} c_2(t, s) \alpha(X) \] for any compact subset \( T \) of \( I_\varepsilon. \)
8. the zero function is the unique continuous solution of the inequality:
\[ p(t) \leq c_1 T \sup_{z \in I} \int_0^T c_2(z, s) p(s) \, ds \] on \( I. \)

Then there exists at least one solution of problem (1).

**Proof** We define the operator \( N : C[0, T] \to C[0, T] \) by
\[ Ny(t) = y_0 + \int_0^t f \left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz. \]
We require that $N : \tilde{B} \to \tilde{B}$ is continuous. Because

(i)

$$
\|Ny(t)\| = \left\| y_0 + \int_0^t f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz \right\|
$$

$$
\leq \|y_0\| + \left\| \int_0^t f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz \right\|
$$

$$
\leq \|y_0\| + \int_0^T \|f(z, y(z), \int_0^z k(z, s, y(s)) \, ds)\| \, dz
$$

$$
\leq \|y_0\| + \int_0^T a(t) \, dt = b
$$

so $Ny(t) \in B$, for $t \in I$.

Now we will show continuity of $N$.

(ii) Let $y_n \to y$ in $C[0, T]$. Then

$$
\|Ny_n - Ny\| = \sup_{t \in [0, T]} \left\| \int_0^t f\left( z, y_n(z), \int_0^z k(z, s, y_n(s)) \, ds \right) \, dz \right\|
$$

$$
- \int_0^t f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz \right\|
$$

$$
\leq \sup_{t \in [0, T]} \left\| \int_0^t \left[ f\left( z, y_n(z), \int_0^z k(z, s, y_n(s)) \, ds \right) \right] \, dz \right\|
$$

$$
- \int_0^t f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz \right\|
$$

$$
\leq \sup_{t \in [0, T]} \left\| \int_0^t \left[ f\left( z, y_n(z), \int_0^z k(z, s, y_n(s)) \, ds \right) \right] \, dz \right\|
$$

$$
- \int_0^t f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz \right\|
$$

$$
\leq \sup_{t \in [0, T]} \left\| \int_0^t \left[ f\left( z, y_n(z), \int_0^z k(z, s, y_n(s)) \, ds \right) \right] \, dz \right\|
$$

$$
- \int_0^t f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz \right\|
$$

$$
+ \sup_{t \in [0, T]} \left\| \int_0^t \left[ f\left( z, y_n(z), \int_0^z k(z, s, y_n(s)) \, ds \right) \right] \, dz \right\|
$$

$$
- \int_0^t f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz \right\|.
$$
Because \( f \) and \( k \) are \( L^1 \)-Carathéodory functions and \( \|y_n - y\| \to 0 \) so \( \|Ny_n - Ny\| \to 0 \).

From (i) and (ii) follows that \( N : \tilde{B} \to \tilde{B} \) is continuous.

Now we will show that the set \( N(B) \) is equicontinuous subset. This follows from inequality:

\[
\|Ny(t) - Ny(\tau)\| = \sup_{t \in [0,T]} \left\| \int_0^t f\left(z, y(z), \int_0^z k(z, s, y(s)) \, ds\right) \, dz \right\|
\leq \sup_{t \in [0,T]} \int_0^t \left\| f\left(z, y(z), \int_0^z k(z, s, y(s)) \, ds\right) \right\| \, dz
\leq \int_0^t a(z) \, dz \quad \text{for every } y \in B.
\]

Observe that the fixed point of the operator \( N \) is the solution of the problems (1) and (1'). Now we prove that fixed point of the operator \( N \) exists using fixed point Theorem 1.

Let \( V \subset \tilde{B} \) be a countable set and \( \tilde{V} = \overline{\text{conv}}(N(V) \cup \{x\}) \). Because \( V \) is an equicontinuous then \( t \mapsto v(t) = \alpha(V(t)) \) is continuous on \( I \). Let \( t \in I \) and \( \varepsilon > 0 \). Using the Lusin’s theorem, there exists a compact subset \( I_\varepsilon \) of \( I \) such that \( \text{mes}(I \setminus I_\varepsilon) \leq \varepsilon \) and a function \( s \mapsto c_2(t, s) \) is continuous on \( I_\varepsilon \). We divide on interval \( I = [0, T] \): \( 0 = t_0 < t_1 < \cdots < t_n = T \), like this

\[
\|c_2(t, s)v(r) - c_2(t, u)v(z)\| < \varepsilon \quad \text{for } s, r, u, z \in T_i = D_i \cap I_\varepsilon,
\]

where \( D_i = [t_{i-1}, t_i], \ i = 1, 2, \ldots, n. \) Let \( V_i = \{u(s): u \in V, s \in D_i\} \).

We notice

\[
\alpha\left(\int_I k(t, s, V(s)) \, ds\right) \leq \alpha\left(\int_{I_\varepsilon} k(t, s, V(s)) \, ds + \int_{I \setminus I_\varepsilon} k(t, s, V(s)) \, ds\right)
\leq \alpha\left(\int_{I_\varepsilon} k(t, s, V(s)) \, ds\right) + \varepsilon_1,
\]

where \( \varepsilon_1 \to 0 \) if \( \varepsilon \to 0 \).
and

\[
\int_I k(z, s, V(s)) \, ds \subset \sum_{i=1}^{n} \int_{T_i} k(z, s, V(s)) \, ds \\
\subset \sum_{i=1}^{n} \operatorname{mes} T_i \overline{\operatorname{conv}} k(z, T_i \times V_i).
\]

Using the properties of measure of noncompactness \(\alpha\) we have

\[
\alpha\left(\int_I k(z, s, V(s)) \, ds\right) \leq \sum_{i=1}^{n} \operatorname{mes} T_i \alpha(k(z, T_i \times V_i)) \\
\leq \sum_{i=1}^{n} \operatorname{mes} T_i \sup_{s \in T_i} c_2(z, s)\alpha(V_i) \\
= \sum_{i=1}^{n} \operatorname{mes} T_i c_2(z, q_i) v(s_i),
\]

where \(q_i \in T_i, s_i \in D_i\).

Moreover, because \(\|c_2(t, s)v(s) - c_2(t, q_i)v(s_i)\| < \varepsilon\) for \(s \in T_i\) we have

\[
\sum_{i=1}^{n} \operatorname{mes} T_i c_2(t, q_i) v(s_i) \\
\leq \sum_{i=1}^{n} \operatorname{mes} T_i \|c_2(t, q_i)v(s_i) - c_2(t, s_i)v(s_i)\| + \sum_{i=1}^{n} \operatorname{mes} T_i c_2(t, s_i)v(s_i) \\
\leq \varepsilon_2 + \sum_{i=1}^{n} \operatorname{mes} T_i c_2(t, s_i)v(s_i),
\]

where \(\varepsilon_2 \to 0\) if \(\varepsilon \to 0\). So

\[
\alpha\left(\int_I k(z, s, y(s)) \, ds\right) \leq \int_{I_\varepsilon} c_2(z, s)v(s) \, ds + \varepsilon_2
\]

then, because \(\varepsilon_2 \to 0\) if \(\varepsilon \to 0\) so

\[
\alpha\left(\int_I k(z, s, y(s)) \, ds\right) \leq \int_I c_2(z, s)v(s) \, ds.
\]
Because $\bar{V} = \overline{\text{conv}}(N(V) \cup \{x\})$, then by the property of measure of noncompactness we have
\begin{align*}
\alpha(V(t)) &= \alpha(\overline{\text{conv}}(N(V(t)) \cup \{x\})) \\
&\leq \alpha(\int_0^t f(z, V(z)), \int_0^z k(z, s, V(s)) \, ds) \, dz \\
&\leq \int_0^t \alpha(f(z, V(z)), \int_0^z k(z, s, V(s)) \, ds) \, dz \\
&\leq \int_0^t c_1 \cdot \max(\alpha(V(z))), \alpha(\int_0^z k(z, s, V(s)) \, ds) \, dz \\
&\leq c_1 \cdot T \cdot \sup_{z \in I} \alpha(\int_0^z k(z, s, V(s)) \, ds) \\
&\leq c_1 \cdot T \cdot \sup_{z \in I} \int_I c_2(z, s) v(s) \, ds.
\end{align*}
So
\begin{align*}
v(t) &\leq c_1 \cdot T \sup_{z \in I} \int_0^T c_2(z, s) v(s) \, ds.
\end{align*}

By (8) we have that $v(t) = \alpha(V(t)) = 0$. Using Arzelá–Ascoli’s theorem we obtain that $V$ is relatively compact. By Theorem 1 the operator $N$ has a fixed point. This means that there exists a solution of problem (1).

**Remark** Theorem 1 extends the existence theorem from Meehan and O’Regan [13] and O’Regan [17].

### 3 AN EXISTENCE RESULT FOR INTEGRODIFFERENTIAL EQUATIONS IN WEAK SENSE

In this part we prove a theorem for the existence of pseudo-solutions to the Cauchy problem
\begin{align*}
y'(t) &= f(t, y(t), \int_0^t k(t, s, y(s)) \, ds), \\
y(0) &= y_0
\end{align*}
in Banach spaces. Functions $f$ and $k$ will be assumed Pettis integrable but this assumption is not sufficient for the existence of solutions. We impose a weak compactness type condition expressed in terms of measures of weak noncompactness. Throughout this part $(E, \| \cdot \|)$ will
denote a real Banach space, $E^*$ the dual space. Unless otherwise stated, we assume that "$\int$" denotes the Pettis integral.

A function $g : E \rightarrow E$ is said to be weakly–weakly sequentially continuous if for each weakly convergent sequence $(x_n) \subset E$, a sequence $(g(x_n))$ is weakly convergent in $E$.

Fix $x^* \in E^*$, and consider the equation

\[(9) \quad (x^* x)'(t) = x^* f(t, x(t), \int_0^t k(t, s, x(s)) \, ds), \quad t \in I.\]

Now, we can introduce the following definition:

**Definition 3** [6,8] A function $x : I \rightarrow E$ is said to be a pseudo-solution of the Cauchy problem (2) if it satisfies the following conditions:

(i) $x(\cdot)$ is absolutely continuous,
(ii) $x(0) = X_0$,
(iii) for each $x^* \in E^*$ there exists an negligible set $A(x^*)$ (i.e. mes $A(x^*) = 0$), such that for each $t \notin A(x^*)$:

\[(x^* x)'(t) = x^* \left( f \left( t, x(t), \int_0^t k(t, s, y(s)) \, ds \right) \right).\]

In other words by a pseudo-solution of (2) we will understand an absolutely continuous function such that $x(0) = X_0$, and $x(\cdot)$ satisfies (2) a.e., for each $x^* \in E^*$.

In this part we use a weak measure of noncompactness of de Blasi’s $\beta$. It is necessary to remark that the following lemma is true:

**Lemma 1** [9,14] Let $\mathcal{H} \subset C_w(I, E)$ be a family of strongly equicontinuous functions. Then the function $t \mapsto \nu(t) = \beta(\mathcal{H}(t))$ is continuous and $\beta(\mathcal{H}(I)) = \sup \{ \beta(\mathcal{H}(t)) : t \in I \}$.

Assume that in addition to (1), (2), (5) and (6),

(10) $k$ is a Carathéodory’s weakly–weakly sequentially continuous function $I^2 \times B$ into $E$;
(11) $f$ is Carathéodory’s weakly–weakly sequentially continuous function from $I \times B \times B$ into $E$;
(12) for any continuous function $y : I \rightarrow E$, functions $k(\cdot, \cdot, y(\cdot))$ and $f(\cdot, y(\cdot), \int_0^\cdot k(\cdot, s, y(s)) \, ds)$ are Pettis integrable.
THEOREM 4  Assume, in addition to (1), (2), (5) and (10–12) that

(13) there exists a constant $c_3$ such that for every interval $J \subseteq I$ and for any subsets $A, C$ of $B$

$$\beta(f(J, A, C)) \leq c_3 \max\{\beta(A), \beta(C)\},$$

(14) there exists an integrable function $c_4 : I \to \mathbb{R}^+$ such that for every $t \in I$, $\varepsilon > 0$ and for every bounded subset $X$ of $B$ there exists a closed subset $I_\varepsilon$ of $I$ such that $\text{mes}(I \setminus I_\varepsilon) < \varepsilon$ and

$$\beta(k(J, J \times X)) \leq \sup_{s \in J} c_4(s) \beta(X), \quad \text{for any } J \subseteq I.$$

Then there exists at least one pseudo-solution of the problem (2).

**Proof**  We define the operator $G : C[0, T] \to C[0, T]$ by

$$Gy(t) = y_0 + \int_0^t f(z, y(z), \int_0^z k(z, s, y(s)) \, ds) \, dz.$$

We require that $G : \tilde{B} \to \tilde{B}$ is weakly sequentially continuous, where

$$\tilde{B} = \left\{ y \in C[0, T] : \|y\| \leq b, \quad b = \|y_0\| + \int_0^T a(t) \, dt \right\}.$$

Because

(i) For any $y^* \in E^*$ such that $\|y^*\| \leq 1$ and for any $y \in B$,

$$\left| y^* \left[ f(z, y(z), \int_0^z k(z, s, y(s)) \, ds) \right] \right|$$

$$\leq \|y^*\| \left\| f(z, y(z), \int_0^z k(z, s, y(s)) \, ds) \right\|$$

$$\leq \left\| f(z, y(z), \int_0^z k(z, s, y(s)) \, ds) \right\| \leq a(z)$$

so

$$|y^* Gy(t)| \leq |y^* y_0| + \int_0^t |y^* \left[ f(z, y(z), \int_0^z k(z, s, y(s)) \, ds) \right] | \, dz$$

$$\leq \|y_0\| + \int_0^t a(t) \, dt \leq \|y_0\| + \int_0^T a(t) \, dt = b.$$
From here

$$\sup \{|y^* Gy(t)|; y^* \in E^*, \|y^*\| \leq 1\} \leq b \quad \text{and} \quad \|Gy(t)\| \leq b$$

so $Gy(t) \in B$.

(ii) Now we will show that set $G(\bar{B})$ is strongly equicontinuous subset.

This follows from the inequality

$$|y^*[Gy(t) - Gy(\tau)]| = \left| y^* \left[ \int_\tau^t f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz \right] \right| \leq \int_\tau^t \left| y^* f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \right| \, dz \leq \int_\tau^t a(z) \, dz.$$

(iii) Now we will show weakly sequentially continuity of $G$.

Let $y_n \to y$ in $(C[0,T], \omega)$.

Then

$$|y^*[Gy_n(t) - Gy(t)]| = \left| y^* \left[ \int_0^t f\left( z, y_n(z), \int_0^z k(z, s, y_n(s)) \, ds \right) \, dz \right] \right| \leq \int_0^t \left| y^* f\left( z, y_n(z), \int_0^z k(z, s, y_n(s)) \, ds \right) \right| \, dz \leq \int_0^T \left| y^* f\left( z, y_n(z), \int_0^z k(z, s, y_n(s)) \, ds \right) \right| \, dz$$

$$= \left| y^* \left[ \int_0^t f\left( z, y(z), \int_0^z k(z, s, y(s)) \, ds \right) \, dz \right] \right|.$$
Because $f$ and $k$ are $L^1$-Carathéodory functions and $y_n \rightarrow y$ in $(C[0, T], \omega)$ so

$$|y^*[Gy_n(t) - Gy(t)]| \rightarrow 0.$$ 

From here

$$\sup\{y^*[Gy_n(t) - Gy(t)]: y^* \in E^*, \|y^*\| \leq 1\} \rightarrow 0.$$ 

From (i) and (iii), follows that $G : \tilde{B} \rightarrow \tilde{B}$ is weakly–weakly sequentially continuous.

Observe that the fixed point of the operator $G$ is the pseudo-solution of the problem

$$y(t) = y_0 + \int_0^t f\left(z, y(z), \int_0^z k(z, s, y(s)) \, ds\right) \, dz.$$ 

(2')

Now we prove that fixed point of the operator $G$ exists using fixed point Theorem 2.

Let $V \subset \tilde{B}$ be a countable set and $\tilde{V} = \overline{\text{conv}}(G(V) \cup \{0\})$. Because $V$ is equicontinuous then $t \rightarrow v(t) = \beta(V(t))$ is continuous on $I$ (by Lemma 1).

Let $t \in I$ and $\varepsilon > 0$. Using the Luzin’s theorem, there exists a compact subset $I_\varepsilon$ of $I$ such that $\text{mes}(I \setminus I_\varepsilon) < \varepsilon$ and a function $s \rightarrow c_4(s)$ is continuous. We divide an interval $I = [0, T]: 0 = t_0 < t_1 < \cdots < t_n = T$, like this $\|c_4(s)v(r) - c_4(u)v(z)\| < \varepsilon$ for $s, r, u, z \in T = \bigcup_{i} I_i$, where $\mathcal{D}_i = [t_{i-1}, t_i]$.

We notice

$$\beta\left(\int_I f\left(z, V(z), \int_0^z k(t, s, V(s)) \, ds\right) \, dz\right)$$

$$\leq \beta\left(\int_{I_\varepsilon} f\left(z, V(z), \int_0^z k(t, s, V(s)) \, ds\right) \, dz\right)$$

$$+ \beta\left(\int_{I \setminus I_\varepsilon} f\left(z, V(z), \int_0^z k(t, s, V(s)) \, ds\right) \, dz\right)$$

$$\leq \beta\left(\int_{I_\varepsilon} f\left(z, V(z), \int_0^z k(t, s, V(s)) \, ds\right) \, dz\right) + \varepsilon'.$$
Using the properties of weak measure of noncompactness $\beta$ we have
\[
\beta\left(\int f\left(z, V(z), \int_0^z k(t, s, V(s)) \, ds\right) \, dz\right)
\leq \beta\left(\sum_{i=1}^n \operatorname{mes} T_i \operatorname{conv} f\left(T_i, V(T_i), \sum_{i=1}^n \operatorname{mes} T_i \operatorname{conv} k(t_i, T_i, V_i)\right)\right)
\leq \sum_{i=1}^n \operatorname{mes} T_i \beta\left(f\left(T_i, V(T_i), \sum_{i=1}^n \operatorname{mes} T_i \operatorname{conv} k(t_i, T_i, V_i)\right)\right)
\leq \sum_{i=1}^n \operatorname{mes} T_i c_3 \cdot \max \beta(V(T_i)), \beta\left(\sum_{i=1}^n \operatorname{mes} T_i \operatorname{conv} k(T_i, T_i, V_i)\right)
\leq \sum_{i=1}^n \operatorname{mes} T_i c_3 \sum_{i=1}^n \operatorname{mes} T_i \beta(k(T_i, T_i, V_i))
\leq T c_3 \sum_{i=1}^n \operatorname{mes} T_i \sup_{s \in T_i} c_4(s) \beta(V_i)
= T c_3 \sum_{i=1}^n \operatorname{mes} T_i c_4(s_i) \beta(V(T_i))
= T c_3 \left[\sum_{i=1}^n \operatorname{mes} T_i c_4(t_i) \beta(V(t_i)) + \sum_{i=1}^n \operatorname{mes} T_i [c_4(s_i) \beta(V(t_i)) - c_4(t_i) \beta(V(t_i))]\right]
\]

From here
\[
\beta\left(\int f\left(z, V(z), \int_0^z k(z, s, V(s)) \, ds\right) \, dz\right)
\leq T c_3 \int_0^t c_4(s) \beta(V(s)) \, ds + \varepsilon_2,
\]

Because $\varepsilon_2 \to 0$ if $\varepsilon \to 0$ we have
\[
\beta(V(t)) \leq \beta(G(V(t)))
\leq \beta\left(\int_0^t f\left(z, y(z), \int_0^z k(z, s, y(s)) \, ds\right) \, dz\right)
\leq T c_3 \int_0^t c_4(s) v(s) \, ds.
\]
So

\[ v(t) \leq Tc_3 \int_0^t c_4(s)\beta(V(s)) \, ds. \]

By Gronwall's inequality we have that \( v(t) = \beta(V(t)) = 0 \).
Using Arzelà–Ascoli's theorem we obtain that \( V \) is weakly relatively compact.

By Theorem 2 the operator \( G \) has a fixed point. This means that there exists a pseudo-solution of problem (2).

Remark Theorem 4 extends the existence theorems from Krzyśka [12], Cichoń [6], O'Regan [16] and others.

References