A Volterra Inequality with the Power Type Nonlinear Kernel

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(Received 20 February 2000; In final form 12 April 2000)

In the paper, we characterize nonnegative, locally integrable functions $k$, for which the nonlinear convolution integral inequality $u(s) \leq k * g(u(s))$, with the power type nonlinearity $g$ has nontrivial solutions.

Keywords and Phrases: Nonlinear Volterra integral equations; Integral inequalities; Trivial solutions; Uniqueness of the solution; The generalized Osgood condition

1991 Mathematics Subject Classifications: 45D05, 45G10, 45M20

1. INTRODUCTION

We study the integral inequality

$$u(x) \leq \int_0^x k(x-s)[u(s)]^\beta \, ds \quad (0 < x, 0 < \beta), \quad (1.1)$$

where $k > 0$ is a given locally integrable function. It is clear that $u(x) \equiv 0$ is a trivial solution of (1.1). Therefore, we are interested further in nontrivial continuous, nonnegative solutions $u$ of (1.1).

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This inequality arises in the study of uniqueness problem for a more general integral equation

\[ y(t) = \int_0^t h(t, s, y(s)) ds + f(t), \quad t \geq 0 \]

in some Banach space. For example, if one considers two solutions \( y_1 \) and \( y_2 \), takes \( x(t) = \|y_1(t) - y_2(t)\| \) and assumes that

\[ \|h(t, s, y_1(s)) - h(t, s, y_2(s))\| \leq k(t - s)\|y_1(s) - y_2(s)\|^{\beta}, \]

then one obtains inequality (1.1) for \( x(t) \).

First, we note that if \( 1 \leq \beta \), then (1.1) has no nontrivial solutions. This due the fact that the integral operator

\[ Tu(x) = \int_0^x k(x - s)[u(s)]^{\beta} ds \quad (\beta \geq 1) \]

is Lipschitz continuous in the class of nonnegative, continuous functions. Therefore, throughout the paper, we assume that \( 0 < \beta < 1 \). It is also important to note that the existence of a nontrivial solution to (1.1) is equivalent to the existence of such a nontrivial solution to the corresponding equation

\[ u(x) = \int_0^x k(x - s)[u(s)]^{\beta} ds \quad (0 < x, 0 < \beta < 1). \quad (1.2) \]

To see this, we consider any nontrivial solution \( v(x) \) of (1.1). To deal with nondecreasing functions, we define

\[ \bar{v}(x) = \sup v(s), \quad 0 \leq s \leq x. \]

Since, the integral operator \( T \) has the following monotonicity properties:

\[ Tw_1(x) \leq Tw_2(x) \quad \text{for any } 0 \leq w_1(x) \leq w_2(x) \]

and

\[ Tw(x) \text{ is nondecreasing for any nondecreasing function } 0 \leq w(x), \]
we easily see that \( \tilde{v}(x) \) is also a nontrivial solution to (1.1). Furthermore, it follows from the inequality

\[
\tilde{v}(x) \leq \int_0^x k(x-s)[\tilde{v}(s)]^\beta ds \leq K(x)[\tilde{v}(x)]^\beta,
\]

where \( K(x) = \int_0^x k(s)ds \) that

\[
\tilde{v}(x) \leq K(x)^{1/(1-\beta)}.
\]

Now, we construct a function sequence

\[
\nu_0(x) = K(x)^{1/(1-\beta)}, \quad \nu_{n+1}(x) = T\nu_n(x), \quad n = 1, 2, \ldots.
\]

We verify directly that \( T\nu_0(x) \leq \nu_0(x) \) and as a consequence of this, we obtain

\[
\nu_{n+1}(x) = T\nu_n(x) \leq \nu_n(x) \quad \text{for } n = 1, 2, \ldots.
\]

Thus \( \{\nu_n(x)\} \) is a nonincreasing sequence of continuous functions. Since

\[
\tilde{v}(x) \leq \nu_0(x) \quad \text{and} \quad \tilde{v}(x) \leq T\tilde{v}(x) \leq T\nu_0(x) \leq \nu_0(x),
\]

we obtain \( \tilde{v}(x) \leq \nu_n(x) \) for \( n = 1, 2, \ldots \). Now, we consider the limit function

\[
u(x) = \lim_{n \to \infty} \nu_n(x) = \lim_{n \to \infty} T\nu_n(x) \geq \tilde{v}(x).
\]

Such a \( u(x) \) is a nontrivial solution of (1.2).

Equation (1.2) is a very special case of the equation

\[
u(x) = \int_0^x k(x-s)g(u(s))ds, \quad (1.3)
\]

where \( g \) is a continuous and nondecreasing function.

There is a wide literature, where the problem of the existence of nontrivial solutions for (1.3) was studied and some necessary and sufficient conditions were given, see [1, 4, 7]. They were formulated in the form of so called the generalized Osgood conditions. One of the most strength results was obtained for the logarithmicly concave
kernels $k$. For example, it is known that for such kernels the following condition

$$\int_0^\delta (K^{-1})' \left( \frac{s}{g(s)} \right) \frac{ds}{g(s)} < \infty,$$

where $K^{-1}$ is inverse to $K$ and $\delta > 0$ is sufficiently small, is necessary for the existence of nontrivial solutions to (1.3). Moreover, in the case $k(x) = x^{\alpha-1}$ or $\exp(-x^{-\alpha})$, $\alpha > 0$ this condition is also sufficient, see [2, 3, 5]. Unfortunately, if $g(u) = u^\beta$, $0 < \beta < 1$ this condition is satisfied for any $k$. On the other hand, it is known that if $k(x) = \exp(-\exp(x^{-\alpha}))$, then Eq. (1.3) has a nontrivial solution if and only if $0 < \alpha < 1$, see [6, 8]. Our aim is to characterize those kernels $k$, for which the inequality (1.1) or equivalently Eq. (1.2) has nontrivial solutions. Our main result is established in the following theorem.

**Theorem** The inequality (1.1) has a nontrivial solution if and only if $0 < \beta < 1$ and

$$\int_0^\delta K^{-1}(s) \frac{ds}{s(-\ln s)} < \infty,$$

where $\delta > 0$ is a sufficiently small number.

**Remark 1** We directly verify that for the kernels $k(x) = \exp(-\exp(x^{-\alpha}))$ mentioned above the following inequalities $k(0.5x) \leq K(x) \leq k(2x)$ hold at the vicinity of zero. Now, we easily see that the condition in Theorem is satisfied in this case, if and only if $0 < \alpha < 1$.

**Remark 2** A substitution $s = \tau^\alpha$ ($0 < \alpha < 1$) into the integral above changes the condition in Theorem to the following

$$\int_0^\delta K^{-1}(\tau^\alpha) \frac{d\tau}{\tau(-\ln \tau)} < \infty.$$

**2. MAIN STEPS OF THE PROOF OF THEOREM**

The necessity part of the theorem. Consider the nontrivial solution $u$ of (1.2) constructed above. We note that Eq. (1.2) has also other nontrivial solutions. For example, the functions $u_c(x) = 0$ for $0 \leq x < c$
and \( u_c(x) = u(x - c) \) for \( x \geq c \) (\( c > 0 \)) are such solutions. Manipulating with \( c \), if necessary we can choose \( u \) such that \( u(0) = 0 \) and \( u(x) > 0 \) for \( x > 0 \). It follows from the construction described above that \( u \) is nondecreasing. Furthermore, the integration by parts gives

\[
u(x) = \int_0^x K(x - \tau)d[u(\tau)\beta], (2.1)\]

from which we infer that \( u \) is absolutely continuous and increasing. Finally, the substitution \( s = u(\tau) \) into integral (2.1) gives

\[
x = \int_0^x K(u^{-1}(x) - u^{-1}(s))d(s^\beta),
\]

where \( u^{-1} \) is inverse to \( u \). Let \( \phi(x) = x^{1/\beta} < x < 1 \). Splitting the integral above into two parts we obtain

\[
x \leq K(u^{-1}(x))\phi(x)^\beta + K(u^{-1}(x) - u^{-1}(\phi(x)))x^\beta. \tag{2.2}
\]

Since \( K(u^{-1}(x)) \to 0 \) as \( x \to 0 \), it follows from (2.2) that

\[
\frac{1}{2} x^{1-\beta} \leq K(u^{-1}(x) - u^{-1}(\phi(x))),
\]

or

\[
K^{-1}\left(\frac{1}{2} x^{1-\beta}\right) \leq u^{-1}(x) - u^{-1}(\phi(x)) \tag{2.3}
\]

for \( 0 < x < \delta \), where \( \delta > 0 \) is sufficiently small.

Now, we note that for any \( 0 < x < \delta \) the sequence

\[
x_0 = x, \quad x_{n+1} = \phi(x_n), \quad n = 1, 2, \ldots
\]

is decreasing and convergent to zero.

Since

\[
\int_{x_{n+1}}^{x_n} K^{-1}\left(\frac{1}{2} s^{1-\beta}\right) \frac{ds}{s(-\ln s)} \leq (-\ln \beta)K^{-1}\left(\frac{1}{2} x_n^{1-\beta}\right),
\]
it follows from (2.3) that
\[ \int_0^x K^{-1}\left(\frac{1}{2}s^{1-\beta}\right) \frac{ds}{s(-\ln s)} < \infty \]
for \(0 < x < \delta\), which gives easily our assertion.

The sufficient part of the theorem. We are going to construct one of the solutions to (1.1). Let \(\psi(x) = x^{2/(1+\beta)} < x < 1\). We expect that the function \(F\) given by its inverse
\[ F^{-1}(x) = \gamma \int_0^x K^{-1}\left(s^{1-\beta/2}\right) \frac{ds}{s(-\ln s)}, \quad \gamma = 1 / \ln(2/(1+\beta)) \]
is such a solution.

First, we note that
\[ \int_0^x K(F^{-1}(x) - F^{-1}(s))d(s^\beta) \]
\[ \geq \int_0^{\psi(x)} K(F^{-1}(x) - F^{-1}(s))d(s^\beta) \]
\[ \geq K(F^{-1}(x) - F^{-1}(\psi(x)))\psi(x)^\beta. \]

We observe also that
\[ F^{-1}(x) - F^{-1}(\psi(x)) = \gamma \int_{\psi(x)}^x K^{-1}\left(s^{1-\beta/2}\right) \frac{ds}{s(-\ln s)} \]
\[ \geq \gamma K^{-1}\left(\psi(x)^{(1-\beta/2)}\right) \int_{\psi(x)}^x \frac{ds}{s(-\ln s)} \]
\[ = K^{-1}\left(\psi(x)^{(1-\beta/2)}\right). \]

It follows from two inequalities above that
\[ \int_0^x K(F^{-1}(x) - F^{-1}(s))d(s^\beta) \geq \psi(x)^{(1+\beta)/2} = x, \]
for \(0 < x < 1\). Now the substitution \(\tau = F(s)\) into the integral above shows that
\[ \int_0^x K(x - s)d(F(\tau)^\beta) \geq F(x). \]
Finally, the integration by parts shows that $F(x)$ satisfies (1.1), which ends the proof.

References


