Research Article

An Analog of the Adjugate Matrix for the Outer Inverse $A_{T,S}^{(2)}$

Xiaoji Liu,¹,² Guanyan Zhu,¹ Guangping Zhou,¹ and Yaoming Yu³

¹ School of Science, Guangxi University for Nationalities, Nanning 530006, China
² Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis, Nanning 530006, China
³ School of Mathematical Sciences, Monash University, Caulfield East, VIC 3800, Australia

Correspondence should be addressed to Xiaoji Liu, liuxiaoji.2003@yahoo.com.cn

Received 5 August 2011; Revised 24 November 2011; Accepted 7 December 2011

Academic Editor: Kui Fu Chen

Copyright © 2012 Xiaoji Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the determinantal representation by exploiting the limiting expression for the generalized inverse $A_{T,S}^{(2)}$. We show the equivalent relationship between the existence and limiting expression of $A_{T,S}^{(2)}$ and some limiting processes of matrices and deduce the new determinantal representations of $A_{T,S}^{(2)}$, based on some analog of the classical adjoint matrix. Using the analog of the classical adjoint matrix, we present Cramer rules for the restricted matrix equation $AXB = D$, $R(X) \subset T$, $A(X) \supset S$.

1. Introduction

Throughout this paper $\mathbb{C}^{m\times n}$ denotes the set of $m \times n$ matrices over the complex number field $\mathbb{C}$, and $\mathbb{C}_{r}^{m\times n}$ denotes its subset in which every matrix has rank $r$. $I$ stands for the identity matrix of appropriate order (dimension).

Let $A \in \mathbb{C}^{m\times n}$, and let $M$ and $N$ be Hermitian positive definite matrices of orders $m$ and $n$, respectively. Consider the following equations:

1. $AXA = A$,
2. $XAX = A$,
3. $(AX)^* = AX$,
\[(MAX)^* = MAX, \quad (3M)\]
\[(XA)^* = XA, \quad (4)\]
\[(NXA)^* = NXA. \quad (4N)\]

X is called a \(\{2\}\) (or outer) inverse of \(A\) if it satisfies (2) and denoted by \(A^{(2)}\). X is called the Moore-Penrose inverse of \(A\) if it satisfies (1), (2), (3), and (5) and denoted by \(A^\dagger\). X is called the weighted Moore-Penrose inverse of \(A\) (with respect to \(M, N\)) if it satisfies (1), (2), (4), and (6) and denoted by \(A^*_{MN}\) (see, e.g., [1, 2]).

Let \(A \in \mathbb{C}^{n \times n}\). Then a matrix \(X\) satisfying
\[
A^kXA = A^k, \quad (1^k)
\]
\[
XAX = X, \quad (2^*)
\]
\[
AX = XA, \quad (5)
\]

where \(k\) is some positive integer, is called the Drazin inverse of \(A\) and denoted by \(A_d\). The smallest positive integer \(k\) such that \(X\) and \(A\) satisfy (7), (8), and (9), then it is called the Drazin index and denoted by \(k = \text{Ind}(A)\). It is clear that \(\text{Ind}(A)\) is the smallest positive integer \(k\) satisfying \(\text{rank}(A^k) = \text{rank}(A^{k+1})\) (see [3]). If \(k = 1\), then \(X\) is called the group inverse of \(A\) and denoted by \(A_g\). As is well known, \(A_g\) exists if and only if \(\text{rank}A = \text{rank}A^2\). The generalized inverses, and in particular Moore-Penrose, group and Drazin inverses, have also been studied in the context of semigroups, rings of Banach and \(C^*\) algebras (see [4-8]).

In addition, if a matrix \(X\) satisfies (1) and (5), then it is called a \(\{1, 5\}\)-inverse of \(A\) and is denoted by \(A^{(1,5)}\).

Let \(A \in \mathbb{C}^{m \times n}\), \(W \in \mathbb{C}^{n \times m}\). Then the matrix \(X \in \mathbb{C}^{m \times n}\) satisfying
\[
(AW)^{k+1}XW = (AW)^k, \quad (1^W)
\]
\[
XWAWX = X, \quad (2W)
\]
\[
AWX = XWA, \quad (5W)
\]

where \(k\) is some nonnegative integer, is called the \(W\)-weighted Drazin inverse of \(A\), and is denoted by \(X = A_{d,W}\) (see [9]). It is obvious that when \(m = n\) and \(W = I_n\), \(X\) is called the Drazin inverse of \(A\).

**Lemma 1.1** (see [1, Theorem 2.14]). Let \(A \in \mathbb{C}^{m \times n}\), and let \(T\) and \(S\) be subspaces of \(\mathbb{C}^n\) and \(\mathbb{C}^m\), respectively, with \(\dim T = \dim S^\perp = t \leq r\). Then \(A\) has a \(\{2\}\)-inverse \(X\) such that \(R(X) = T\) and \(\mathcal{N}(X) = S\) if and only if
\[
AT \oplus S = \mathbb{C}^m \quad (1.1)
\]
in which case \(X\) is unique and denoted by \(A^{(2)}_{T,S}\).
If $A_{T,S}^{(2)}$ exists and there exists a matrix $G$ such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$, then $GAA_{T,S}^{(2)} = G$ and $A_{T,S}^{(2)}AG = G$.

It is well known that several important generalized inverses, such as the Moore-Penrose inverse $A^\dagger$, the weighted Moore-Penrose inverse $A_{M,N}$, the Drazin inverse $A_d$, and the group inverse $A_g$, are outer inverses $A_{T,S}^{(2)}$ for some specific choice of $T$ and $S$, are all the generalized inverse $A_{T,S}^{(2)}$, [2]-(or outer) inverse of $A$ with the prescribed range $T$ and null space $S$ (see [2, 10] in the context of complex matrices and [11] in the context of semigroups).

Determinantal representation of the generalized inverse $A_{T,S}^{(2)}$ was studied in [12, 13]. We will investigate further such representation by exploiting the limiting expression for $A_{T,S}^{(2)}$. The paper is organized as follows. In Section 2, we investigate the equivalent relationship between the existence of $A_{T,S}^{(2)}$ and the limiting process of matrices $\lim_{\lambda \to 0} G(AG + \lambda I)^{-1}$ or $\lim_{\lambda \to 0} (GA + \lambda I)^{-1}G$ and deduce the new determinantal representations of $A_{T,S}^{(2)}$, based on some analog of the classical adjoint matrix, by exploiting limiting expression. In Section 3, using the analog of the classical adjoint matrix in Section 2, we present Cramer rules for the restricted matrix equation $AXB = D$, $\mathcal{R}(X) \subseteq T$, $\mathcal{N}(X) \supseteq S$. In Section 4, we give an example for solving the solution of the restricted matrix equation by using our expression.

We introduce the following notations.

For $1 \leq k \leq n$, the symbol $Q_{k,n}$ denotes the set $\{\alpha : \alpha = (a_1, \ldots, a_k), 1 \leq a_1 < \cdots < a_k \leq n, \text{where } a_i, \ i = 1, \ldots, k, \text{ are integers}\}$. And $Q_{k,n}\{j\} := \{\beta : \beta \in Q_{k,n}, j \in \beta\}$, where $j \in \{1, \ldots, n\}$.

Let $A = (a_{ij}) \in \mathbb{C}^{m \times n}$. The symbols $a_j$ and $a_i^*$ stand for the $j$th column and the $i$th row of $A$, respectively. In the same way, denote by $a_j^*$ and $a_i^*$ the $j$th column and the $i$th row of Hermitian adjoint matrix $A^*$. The symbol $A_j(b)$ (or $A_i(b)$) denotes the matrix obtained from $A$ by replacing its $j$th column (or row) with some vector $b$ (or $b^\dagger$). We write the range of $A$ by $\mathcal{R}(A) = \{Ax : x \in \mathbb{C}^n\}$ and the null space of $A$ by $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$. Let $B \in \mathbb{C}^{p \times q}$. We define the range of a pair of $A$ and $B$ as $\mathcal{R}(A,B) = \{AWB : W \in \mathbb{C}^{n \times p}\}$.

Let $a \in Q_{k,m}$ and $\beta \in Q_{k,n}$, where $1 \leq k \leq \min\{m, n\}$. Then $|A_{\beta}^a|$ denotes a minor of $A$ determined by the row indexed by $a$ and the columns indexed by $\beta$. When $m = n$, the cofactor of $a_{ij}$ in $A$ is denoted by $\partial|A|/\partial a_{ij}$.

2. Analogs of the Adjugate Matrix for $A_{T,S}^{(2)}$

We start with the following theorem which reveals the intrinsic relation between the existence of $A_{T,S}^{(2)}$ and of $\lim_{\lambda \to 0} G(AG + \lambda I)^{-1}$ or $\lim_{\lambda \to 0} (GA + \lambda I)^{-1}G$. Here $\lambda \to 0$ means $\lambda \to 0$ through any neighborhood of 0 in $\mathbb{C}$ which excludes the nonzero eigenvalues of a square matrix. In [14], Wei pointed out that the existence of $A_{T,S}^{(2)}$ implies the existence of $\lim_{\lambda \to 0} G(AG + \lambda I)^{-1}$ or $\lim_{\lambda \to 0} (GA + \lambda I)^{-1}G$. The following result will show that the converse is true under some condition.

Theorem 2.1. Let $A \in \mathbb{C}_*^{m \times n}$, and let $T$ and $S$ be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, with $\dim T = \dim S^\perp = t \leq n$. Let $G \in \mathbb{C}_*^{p \times m}$ with $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. Then the following statements are equivalent:

(i) $A_{T,S}^{(2)}$ exists;

(ii) $\lim_{\lambda \to 0} G(AG + \lambda I)^{-1}$ exists and $\text{rank}(AG) = \text{rank}(G)$;

(iii) $\lim_{\lambda \to 0} (GA + \lambda I)^{-1}G$ exists and $\text{rank}(GA) = \text{rank}(G)$. 

In this case,

\[ A_{T,S}^{(2)} = \lim_{\lambda \to 0} (GA + \lambda I)^{-1}G = \lim_{\lambda \to 0} G(AG + \lambda I)^{-1}. \]  

(2.1)

Proof. (i)⇔(ii) Assume that \( A_{T,S}^{(2)} \) exists. By [14, Theorem 2.4], \( \lim_{\lambda \to 0} G(AG + \lambda I)^{-1} \) exists. Since \( G = A_{T,S}AG \), \( \text{rank}(AG) = \text{rank}(G) \).

Conversely, assume that \( \lim_{\lambda \to 0} G(AG + \lambda I)^{-1} \) exists and \( \text{rank}(AG) = \text{rank}(G) \). So

\[ \lim_{\lambda \to 0} (AG + \lambda I)^{-1}AG = \lim_{\lambda \to 0} AG(AG + \lambda I)^{-1} \]

(2.2)

exists. By [15, Theorem], \( (AG) \) exists. So \( (AG)^{(1,5)} \) exists, and then, by [13, Theorem 2], \( A_{T,S}^{(2)} \) exists.

Similarly, we can show that (i)⇔(iii). Equation (2.1) comes from [14, equation (2.16)].

Lemma 2.2. Let \( A = (a_{ij}) \in \mathbb{C}^{m \times n} \) and \( G = (g_{ij}) \in \mathbb{C}^{n \times m} \). Then \( \text{rank}(GA) \leq t \), where \( 1 \leq i \leq n, 1 \leq j \leq m \), and \( \text{rank}(AG) \leq t \), where \( 1 \leq i \leq m, 1 \leq j \leq n \).

Proof. Let \( P_k(a) \) be an \( n \times n \) matrix with \( a \) in the \((i, k)\) entry, 1 in all diagonal entries, and 0 in others. It is an elementary matrix and

\[
(GA)_{ij}(g_{ij}) \prod_{k \neq i} P_k(-a_{jk}) = \left( \begin{array}{ccc}
\sum_{k \neq j} g_{k1}a_{k1} & \cdots & g_{1j} & \cdots & \sum_{k \neq j} g_{kn}a_{kn} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\sum_{k \neq j} g_{nk}a_{k1} & \cdots & g_{nj} & \cdots & \sum_{k \neq j} g_{nk}a_{kn}
\end{array} \right)_{ij} \cdot \left( \begin{array}{ccc}
a_{i1} & \cdots & 0 & \cdots & a_{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
a_{m1} & \cdots & 0 & \cdots & a_{nm}
\end{array} \right)_{ji} \tag{2.3}
\]

It follows from the invertibility of \( P_k(a), i \neq k \), that \( \text{rank}(GA) \leq t \).

Analogously, the inequation \( \text{rank}(AG) \leq t \) can be proved. So the proof is complete.

Recall that if \( f_A(\lambda) = \det(\lambda I + A) = \lambda^n + d_1\lambda^{n-1} \cdots + d_{n-1}\lambda + d_n \) is the characteristic polynomial of an \( n \times n \) matrix—\( A \) over \( \mathbb{C} \), then \( d_i \) is the sum of all \( i \times i \) principal minors of \( A \), where \( i = 1, \ldots, n \) (see, e.g., [16]).
**Theorem 2.3.** Let \( A, T, S, \) and \( G \) be the same as in Theorem 2.1. Write \( G = (g_{ij}) \). Suppose that the generalized inverse \( A_{T,S}^{(2)} \) of \( A \) exists. Then \( A_{T,S}^{(2)} \) can be represented as follows:

\[
A_{T,S}^{(2)} = \left( \frac{x_{ij}}{d_i(GA)} \right)_{n \times m},
\]

where

\[
x_{ij} = \sum_{\beta \in Q_{i,m}(j)} \left| \left( (GA)_j(g_{ij}) \right)_{\beta} \right|, \quad d_i(GA) = \sum_{\beta \in Q_{i,n}} \left| (GA)_{\beta} \right|.
\]

or

\[
A_{T,S}^{(2)} = \left( \frac{y_{ij}}{d_i(AG)} \right)_{n \times m},
\]

where

\[
y_{ij} = \sum_{\alpha \in Q_{i,m}(j)} \left| \left( (AG)_j(g_{ij}) \right)_{\alpha} \right|, \quad d_i(AG) = \sum_{\alpha \in Q_{i,n}} \left| (AG)_{\alpha} \right|.
\]

**Proof.** We will only show the representation (2.5) since the proof of (2.7) is similar. If \(-\lambda\) is not the eigenvalue of \( GA \), then the matrix \( \lambda I + GA \) is invertible, and

\[
(\lambda I + GA)^{-1} = \frac{1}{\det(\lambda I + GA)} \begin{pmatrix}
X_{11} & X_{21} & \cdots & X_{n1} \\
X_{12} & X_{22} & \cdots & X_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
X_{1n} & X_{2n} & \cdots & X_{nn}
\end{pmatrix},
\]

where \( X_{ij}, \ i, j = 1, \ldots, n, \) are cofactors of \( \lambda I + GA \). It is easy to see that

\[
\sum_{i=1}^{n} X_{ii} g_{ij} = \det(\lambda I + GA)_i(g_{ij}).
\]

So, by (2.1),

\[
A_{T,S}^{(2)} = \lim_{\lambda \to 0} \frac{\det(\lambda I + GA)_1(g_{1})}{\det(\lambda I + GA)} \cdots \frac{\det(\lambda I + GA)_1(g_{m})}{\det(\lambda I + GA)} \cdots \frac{\det(\lambda I + GA)_n(g_{1})}{\det(\lambda I + GA)} \cdots \frac{\det(\lambda I + GA)_n(g_{m})}{\det(\lambda I + GA)}.
\]

We have the characteristic polynomial of \( GA \)

\[
f_{GA}(\lambda) = \det(\lambda I + GA) = \lambda^n + d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \cdots + d_n,
\]
where \( d_i \) \((1 \leq i \leq n)\) is a sum of \( i \times i \) principal minors of \( GA \). Since \( \text{rank}(GA) \leq \text{rank}(G) = t \), \( d_n = d_{n-1} = \cdots = d_{t+1} = 0 \) and
\[
\det(\lambda I + GA) = \lambda^n + d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \cdots + d_t \lambda^{n-t}.
\] (2.12)

Expanding \( \det(\lambda I + GA)_i(g_{ij}) \), we have
\[
\det(\lambda I + GA)_i(g_{ij}) = x_1^{(ij)} \lambda^{n-1} + x_2^{(ij)} \lambda^{n-2} + \cdots + x_n^{(ij)} ,
\] (2.13)

where \( x_k^{(ij)} = \sum_{\beta \in Q_k, \alpha \in [i]} |((GA)_i(g_{ij}))_{\beta}^{(i)}| \), \( 1 \leq k \leq n \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).

By Lemma 2.2, \( \text{rank}(GA)_i(g_{ij}) \leq t \) and so \(|((GA)_j(g_{ij}))_{\beta}^{(j)}| = 0 \), \( k > t \) and \( \beta \in Q_k, n \{i\} \), for all \( i, j \). Therefore, \( x_k^{(ij)} = 0 \), \( k \leq n \), for all \( i, j \). Consequently,
\[
\det(\lambda I + GA)_i(g_{ij}) = x_1^{(ij)} \lambda^{n-1} + x_2^{(ij)} \lambda^{n-2} + \cdots + x_n^{(ij)} \lambda^{n-t}.
\] (2.14)

Substituting (2.12) and (2.14) into (2.10) yields
\[
A_{T,S}^{(2)} = \lim_{\lambda \to 0} \frac{1}{\lambda^n} \begin{pmatrix}
\frac{x_1^{(11)} \lambda^{n-1} + \cdots + x_t^{(1m)} \lambda^{n-t}}{d_1} & \cdots & \frac{x_1^{(1m)} \lambda^{n-1} + \cdots + x_t^{(1m)} \lambda^{n-t}}{d_1} \\
\cdots & \cdots & \cdots \\
\frac{x_1^{(n1)} \lambda^{n-1} + \cdots + x_t^{(n1)} \lambda^{n-t}}{d_1} & \cdots & \frac{x_1^{(nm)} \lambda^{n-1} + \cdots + x_t^{(nm)} \lambda^{n-t}}{d_1}
\end{pmatrix}
\] (2.15)

Substituting \( x_{ij} \) for \( x^{(ij)}_i \) in the above equation, we reach (2.5).

**Remark 2.4.** The proofs of Lemma 2.2 and Theorem 2.3 are based on the general techniques and methods obtained previously by [17], respectively.

**Remark 2.5.** (i) By using (2.5), we can obtain (2.17) in [12, Theorem 2.3]. In fact, \( u = d_i(GA) \) and, by the Binet-Cauchy formula,
\[
x_{ij} = \sum_{\beta \in Q_{i,j}} \left| (GA)_{\beta}^{(i)} \right| = \sum_{\beta \in Q_{i,j}} \sum_k \frac{\partial |(GA)_{\beta}^{(i)}|}{\partial s_{ki}} = \sum_{\beta \in Q_{i,j}} \sum_k \frac{\partial |G_{\beta}^{(i)}|}{\partial g_{kj}} \frac{\partial A_{\beta}^{(i)}}{\partial a_{ji}}
\]
\[
= \sum_{\beta \in Q_{i,j}} \sum_{\alpha \in Q_{i,j}} \frac{\partial |A_{\beta}^{(i)}|}{\partial a_{ji}}
\]
(2.16)
where \( s_{kj} = (GA)_{kj} \). Note that \( \frac{\partial |A^\beta_i|}{\partial a_{ij}} = 0 \) if \( i \notin \alpha \) or \( j \notin \beta \). In addition, using the symbols in [13], we can rewrite (2.5) as [13, equation (13)] over \( \mathbb{C} \).

(ii) This method is especially efficient when \( GA \) or \( AG \) is given (comparing with [12, Theorem 2]).

Observing the particular case from Theorem 2.3, \( G = (g_{ij}) = N^{-1} A^* M \), where \( M \) and \( N \) are Hermitian positive definite matrices, we obtain the following corollary in which the symbols \( g_j := (g)_{j} \) and \( g_i := (g)_{i} \).

**Corollary 2.6.** Let \( A \in \mathbb{C}_{r \times n}^{m \times n} \) and \( G = N^{-1} A^* M \), where \( M \) and \( N \) are Hermitian positive definite matrices of order \( m \) and \( n \), respectively. Then

\[
A_{MN}^+ = \left( \frac{x_{ij}}{d_r(GA)} \right)_{n \times m}, \tag{2.17}
\]

where

\[
x_{ij} = \sum_{\rho \in Q_\rho \ast \alpha} \left\| (GA)_{\rho \ast \alpha} (g_{ij})_{\rho \ast \alpha} \right\|_eta, \quad d_r(GA) = \sum_{\rho \in Q_\rho \ast \alpha} \left\| (GA)^{\beta \ast \beta} \right\|_\rho.
\]

or

\[
A_{MN}^+ = \left( \frac{y_{ij}}{d_r(AG)} \right)_{n \times m}, \tag{2.19}
\]

where

\[
y_{ij} = \sum_{\alpha \in Q_\alpha \ast \beta} \left\| (AG)_{\alpha \ast \beta} (g_{ij})_{\alpha \ast \beta} \right\|_\alpha, \quad d_r(AG) = \sum_{\alpha \in Q_\alpha \ast \beta} \left\| (AG)^{\alpha \ast \alpha} \right\|_\beta.
\]

If \( M \) and \( N \) are identity matrices, then we can obtain the following result.

**Corollary 2.7** (see [17, Theorem 2.2]). The Moore-Penrose inverse \( A^+ \) of \( A = (a_{ij}) \in \mathbb{C}_{r \times n}^{m \times n} \) can be represented as follows:

\[
A^+ = \left( \frac{x_{ij}}{d_r(A^* A)} \right)_{n \times m}, \tag{2.21}
\]

where

\[
x_{ij} = \sum_{\rho \in Q_\rho \ast \alpha} \left\| (A^* A)_{\rho \ast \alpha} (a_{ij})_{\rho \ast \alpha} \right\|_\beta, \quad d_r(A^* A) = \sum_{\rho \in Q_\rho \ast \alpha} \left\| (A^* A)^{\beta \ast \beta} \right\|_\rho.
\]

or

\[
A^+ = \left( \frac{y_{ij}}{d_r(AA^*)} \right)_{n \times m}, \tag{2.23}
\]
where

\[ y_{ij} = \sum_{a \in Q_{n,1}} \left| \left( (AA^*)^x_i (a^*_x) \right)_a \right|, \quad d_r(AA^*) = \sum_{a \in Q_{n,1}} \left| (AA^*)_a \right|. \] (2.24)

Note that \( A_{d,W} = (WAW)^{(2)}_{k((AW)^k, A), A}((AW)^k A)_A \). Therefore, when \( G = (AW)^k A \) in Theorem 2.3, we have the following corollary.

**Corollary 2.8.** Let \( A \in \mathbb{C}^{m \times n}, W \in \mathbb{C}^{n \times m}, \) and \( k = \max \{ \text{Ind}(AW), \text{Ind}(WA) \} \). If \( \text{rank} (AW)^k = t, \text{rank} (WA)^k = r, \) and \( \text{rank} (AW)^k A = (c_{ij})_{m \times n} \), then

\[ A_{d,W} = \left( \frac{x_{ij}}{d_t((AW)^{k+2})} \right)_{m \times n} \] (2.25)

where

\[ x_{ij} = \sum_{\beta \in Q_{n,1}} \left| \left( (AW)^{k+2} \right)_i (c_j)^{\beta^r} \right|, \quad d_t((AW)^{k+2}) = \sum_{\beta \in Q_{n,1}} \left| (AW)^{k+2} \right| \] (2.26)

or

\[ A_{d,W} = \left( \frac{y_{ij}}{d_r((WA)^{k+2})} \right)_{m \times n} \] (2.27)

where

\[ y_{ij} = \sum_{a \in Q_{n,1}} \left| \left( (WA)^{k+2} \right)_i (a_j^r) \right|, \quad d_r((WA)^{k+2}) = \sum_{a \in Q_{n,1}} \left| (WA)^{k+2} \right| \] (2.28)

When \( G = A^k \) with \( k = \text{Ind}(A) \) in Theorem 2.3, we have the following corollary.

**Corollary 2.9** (see [17, Theorem 3.3]). Let \( A \in \mathbb{C}^{m \times n} \) with \( \text{Ind} A = k \) and \( \text{rank} A^k = r \), and \( A^k = (a_{ij}^{(k)})_{n \times n} \). Then

\[ A_d = \left( \frac{x_{ij}}{d_r(A^{k+1})} \right)_{n \times n} \] (2.29)

where

\[ d_{ij} = \sum_{\beta \in Q_{n,1}} \left| \left( (A^{k+1}) \left( a_{ij}^{(k)} \right) \right)^{\beta^r} \right|, \quad d_r(A^{k+1}) = \sum_{\beta \in Q_{n,1}} \left| (A^{k+1})^{\beta^r} \right| \] (2.30)
Finally, we turn our attention to the two projectors $A^{(2)}_{T,S}A$ and $AA^{(2)}_{T,S}$. The limiting expressions for $A^{(2)}_{T,S}$ in (2.1) bring us the following:

$$A^{(2)}_{T,S}A = \lim_{\lambda \to 0} (GA + \lambda I)^{-1}GA,$$

$$AA^{(2)}_{T,S} = \lim_{\lambda \to 0} AG(AG + \lambda I)^{-1}. \tag{2.31}$$

**Corollary 2.10.** Let $A, T, S$, and $G$ be the same as in Theorem 2.1. Write $GA = (s_{ij})$ and $AG = (h_{ij})$. Suppose that $A^{(2)}_{T,S}$ exists. Then $A^{(2)}_{T,S}A$ of $AA^{(2)}_{T,S}$ can be represented as follows:

$$A^{(2)}_{T,S}A = \left( \frac{x_{ij}}{d_t(GA)} \right)_{n \times n}, \tag{2.32}$$

where

$$x_{ij} = \sum_{\beta \in Q_{a}(i)} \left| (GA)_{j}(s_{j}) \right|^\beta, \quad d_t(GA) = \sum_{\beta \in Q_{a}} \left| (GA)_{j} \right|^\beta,$$

$$AA^{(2)}_{T,S} = \left( \frac{y_{ij}}{d_t(AG)} \right)_{m \times m}, \tag{2.33}$$

where

$$y_{ij} = \sum_{\alpha \in Q_{m}(j)} \left| (AG)_{j}(h_{i}) \right|^\alpha, \quad d_t(AG) = \sum_{\alpha \in Q_{m}} \left| (AG)_{j} \right|^\alpha.$$

**3. Cramer Rules for the Solution of the Restricted Matrix Equation**

The restricted matrix equation problem is mainly to find solution of a matrix equation or a system of matrix equations in a set of matrices which satisfy some constraint conditions. Such problems play an important role in applications in structural design, system identification, principal component analysis, exploration, remote sensing, biology, electricity, molecular spectroscopy, automatics control theory, vibration theory, finite elements, circuit theory, linear optimal, and so on. For example, the finite-element static model correction problem can be transformed to solve some constraint condition solution and its best approximation of the matrix equation $AX = B$. The undamped finite-element dynamic model correction problem can be attributed to solve some constraint condition solution and its best approximation of the matrix equation $A^TXA = B$. These motivate the gradual development of theory in respect of the solution to the restricted matrix equation in recent years (see [18–27]).

In this section, we consider the restricted matrix equation

$$AXB = D, \quad R(X) \subset T, \quad \mathcal{N}(X) \supset \tilde{S}, \tag{3.1}$$
where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$, $D \in \mathbb{C}^{m \times q}$, $T \subset \mathbb{C}^n$, $S \subset \mathbb{C}^m$, $\tilde{T} \subset \mathbb{C}^d$ and $\tilde{S} \subset \mathbb{C}^p$, satisfy
\[
\dim(T) = \dim(S) = t \leq r, \quad \dim(\tilde{T}) = \dim(\tilde{S}) = \tilde{t} \leq \tilde{r}.
\] (3.2)

Assume that there exist matrices $G \in \mathbb{C}^{nxm}$ and $\tilde{G} \in \mathbb{C}^{n \times p}$ satisfying
\[
\mathcal{R}(G) = T, \quad \mathcal{N}(G) = S, \quad \mathcal{R}(\tilde{G}) = \tilde{T}, \quad \mathcal{N}(\tilde{G}) = \tilde{S}.
\] (3.3)

If $A_{T,S}^{(2)}$ and $B_{T,S}^{(2)}$ exist and $D \in \mathcal{R}(AG, \tilde{G}B)$, then the restricted matrix equation (3.1) has the unique solution
\[
X = A_{T,S}^{(2)}DB_{T,S}^{(2)}
\] (3.4)
(see [2, Theorem 3.3.3] for the proof).

In particular, when $D$ is a vector $b$ and $B = \tilde{G} = I_1$, the restricted matrix equation (3.1) becomes the restricted linear equation
\[
Ax = b, \quad x \in T.
\] (3.5)

If $b \in \mathcal{AR}(G)$, then $x = A_{T,S}^{(2)}b$ is the unique solution of the restricted linear equation (3.5) (see also [10, Theorem 2.1]).

**Theorem 3.1.** Given $A, B, D = (d_{ij}), G = (g_{ij}), \tilde{G} = (\tilde{g}_{ij}), T, S, \tilde{T},$ and $\tilde{S}$ as above. If $A_{T,S}^{(2)}$ and $B_{T,S}^{(2)}$ exist and $D \in \mathcal{R}(AG, \tilde{G}B)$, then $X = A_{T,S}^{(2)}DB_{T,S}^{(2)}$ is the unique solution of the restricted matrix equation (3.1) and it can be represented as
\[
x_{ij} = \frac{\sum_{k=1}^{m} \sum_{\beta \in Q_{a(i)}, \alpha \in Q_{a(j)}} ((GA)_{i,j}(g_{ik}))^{\beta}_{\beta} \left(\left((BG)_{j,k}(\tilde{g}_{ik})\right)_{\alpha}^{\alpha}\right)}{d_i(GA) \ d_i(B\tilde{G})}
\] (3.6)
or
\[
x_{ij} = \frac{\sum_{k=1}^{d} \sum_{\beta \in Q_{a(i)}, \alpha \in Q_{a(j)}} ((GA)_{i,j}(f_{ik}))^{\beta}_{\beta} \left(\left((BG)_{j,k}(\tilde{g}_{ik})\right)_{\alpha}^{\alpha}\right)}{d_i(GA) \ d_i(B\tilde{G})}.
\] (3.7)

where $\tilde{f}_k = d_k \tilde{G}$ and $f_k = Gd_k$, $i = 1, \ldots, n$, and $j = 1, \ldots, p.$
Proof. By the argument above, we have $X = A^{(2)}_{T,S}DB^{(2)}_{\tilde{T},\tilde{S}}$ is the unique solution of the restricted matrix equation (3.1). Setting $Y = DB^{(2)}_{\tilde{T},\tilde{S}}$ and using (2.7), we get that

$$y_{kj} = \sum_{h=1}^{q} d_{kh} \left( B^{(2)}_{\tilde{T},\tilde{S}} \right)_{hj} = \sum_{h=1}^{q} \sum_{a \in Q_{t,n,\{j\}}} \left| \left( \left( B\tilde{G} \right)_{j} \left( \tilde{G}h_\alpha \right) \right)^{a} \right| \frac{\sum_{a \in Q_{t,n,\{j\}}} \left( \left( B\tilde{G} \right)_{j} \left( \tilde{f}_k \right) \right)^{a}}{d_t \left( B\tilde{G} \right)}.$$  

(3.8)

where $\tilde{f}_k = d_k \tilde{G}$. Since $X = A^{(2)}_{T,S}Y$, by (2.5),

$$x_{ij} = \sum_{k=1}^{m} A^{(2)}_{T,S} \left( A^{(2)}_{T,S} \right)_{ik} y_{kj} = \sum_{k=1}^{m} \sum_{\beta \in Q_{t,n,\{i\}}} \left| \left( \left( GA \right)_{j} \left( \tilde{G}h_\beta \right) \right)^{\beta} \right| \frac{\sum_{\beta \in Q_{t,n,\{i\}}} \left( \left( B\tilde{G} \right)_{j} \left( \tilde{f}_k \right) \right)^{\beta}}{d_t \left( B\tilde{G} \right)}.$$  

(3.9)

Hence, we have (3.6).

We can obtain (3.7) in the same way. □

In particular, when $D$ is a vector $b$ and $B = \tilde{G} = I_1$ in the above theorem, we have the following result from (3.7).

**Theorem 3.2.** Given $A, G, T,$ and $S$ as above. If $b \in A\mathcal{R}(G)$, then $x = A^{(2)}_{T,S}b$ is the unique solution of the restricted linear equation $Ax = b$, $x \in T$, and it can be represented as

$$x_i = \frac{\sum_{\beta \in Q_{t,n,\{i\}}} \left| \left( \left( GA \right)_{j} \left( f \right) \right)^{\beta} \right|}{d_t \left( GA \right)}, \quad j = 1, \ldots, n.$$  

(3.10)

where $f = Gb$.

**Remark 3.3.** Using the symbols in [13], we can rewrite (3.10) as [13, equation (27)].
4. Example

Let

\[
A = \begin{pmatrix}
1 & 2 & 2 & 2 \\
0 & 2 & 1 & 1 \\
0 & 0 & 4 & 2 \\
0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix},
B = \begin{pmatrix}
2 & 1 \\
1 & 0
\end{pmatrix},
D = \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix},
G = \begin{pmatrix}
1 & -1 & 2 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\tilde{G} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0
\end{pmatrix}.
\]

(4.1)

Obviously, \( \text{rank } A = 3 \), \( \text{dim } AT = \text{dim } T = 2 \), and

\[
T = \mathcal{R}(G) \subset \mathbb{C}^4, \quad S = \mathcal{N}(G) \subset \mathbb{C}^5, \quad \tilde{T} = \mathcal{R}(\tilde{G}) \subset \mathbb{C}^2, \quad \tilde{S} = \mathcal{N}(\tilde{G}) \subset \mathbb{C}^3.
\]

(4.2)

It is easy to verify that \( AT \oplus S = \mathbb{C}^5 \) and \( B\tilde{T} \oplus \tilde{S} = \mathbb{C}^3 \). Thus, \( A_{T,S}^{(2)} \) and \( B_{\tilde{T},\tilde{S}}^{(2)} \) exist by Lemma 1.1.

Now consider the restricted matrix equation

\[
AXB = D, \quad \mathcal{R}(X) \subset T, \quad \mathcal{N}(X) \supset \tilde{S}.
\]

(4.3)

Clearly,

\[
AG = \begin{pmatrix}
1 & 3 & 4 & 0 & 0 \\
0 & 4 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \tilde{G}B = \begin{pmatrix}
2 & 1 \\
2 & 0
\end{pmatrix},
\]

(4.4)

and it is easy to verify that \( \mathcal{R}(D) \subset \mathcal{R}(AG) \) and \( \mathcal{N}(D) \supset \mathcal{N}(\tilde{G}B) \) hold.

Note that \( \mathcal{R}(D) \subset \mathcal{R}(AG) \) and \( \mathcal{N}(D) \supset \mathcal{N}(\tilde{G}B) \) if and only if \( D \in \mathcal{R}(AG, \tilde{G}B) \). So, by Theorem 3.1, the unique solution of (4.3) exists.
Table 1

<table>
<thead>
<tr>
<th>$y_{ik}$</th>
<th>$i = 1$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 1$</td>
<td>4</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$k = 2$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>$z_{kj}$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 1$</td>
<td>0</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>$k = 2$</td>
<td>-2</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

Computing

$$GA = \begin{pmatrix} 1 & 0 & 9 & 5 \\ 0 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B\tilde{G} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

$$d_2(GA) = \sum_{i} \begin{pmatrix} 1 \\ 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 9 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 4,$$

$$d_2(B\tilde{G}) = \sum_{j} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} = -2, \quad (4.5)$$

$$\tilde{f} = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and setting $y_{ik} = \sum_{\beta \in Q_{\gamma_{x_{ij}}} i}} ((GA)_{i}(\tilde{f}_{k})_{\beta})_{\beta}$, we have Table 1.

Similarly, setting $z_{kj} = \sum_{\alpha \in Q_{\gamma_{x_{ij}}} j}} ((B\tilde{G})_{j}(\tilde{g}_{k})_{\alpha})_{\alpha}$, we have Table 2.

So, by (3.7), we have

$$X = (x_{ij}) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.6)$$

**Acknowledgments**

The authors would like to thank the referees for their helpful comments and suggestions. This work was supported by the National Natural Science Foundation of China 11061005, the Ministry of Education Science and Technology Key Project under Grant 210164, and Grants HCIC201103 of Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis Open Fund.
References
