Research Article

Periodic Boundary Value Problems for Semilinear Fractional Differential Equations

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We study the periodic boundary value problem for semilinear fractional differential equations in an ordered Banach space. The method of upper and lower solutions is then extended. The results on the existence of minimal and maximal mild solutions are obtained by using the characteristics of positive operators semigroup and the monotone iterative scheme. The results are illustrated by means of a fractional parabolic partial differential equations.

1. Introduction

In this paper, we consider the periodic boundary value problem (PBVP) for semilinear fractional differential equation in an ordered Banach space X,

\[ D^\alpha u(t) + Au(t) = f(t, u(t)), \quad t \in I, \]
\[ u(0) = u(\omega), \]

where \( D^\alpha \) is the Caputo fractional derivative of order \( 0 < \alpha < 1 \), \( I = [0, \omega] \), \(-A : D(A) \subset X \to X\) is the infinitesimal generator of a \( C_0\)-semigroup (i.e., strongly continuous semigroup) \( \{T(t)\}_{t \geq 0} \) of uniformly bounded linear operators on \( X \), and \( f : I \times X \to X \) is a continuous function.

Fractional calculus is an old mathematical concept dating back to the 17th century and involves integration and differentiation of arbitrary order. In a later dated 30th of September 1695, L’Hospital wrote to Leibniz asking him about the differentiation of order \( 1/2 \). Leibniz’ response was “an apparent paradox from which one day useful consequences will be drawn.” In the following centuries, fractional calculus developed significantly within
pure mathematics. However, the applications of fractional calculus just emerged in the last few decades. The advantage of fractional calculus becomes apparent in science and engineering. In recent years, fractional calculus attracted engineers’ attention, because it can describe the behavior of real dynamical systems in compact expressions, taking into account nonlocal characteristics like infinite memory [1–3]. Some instances are thermal diffusion phenomenon [4], botanical electrical impedances [5], model of love between humans [6], the relaxation of water on a porous dyke whose damping ratio is independent of the mass of moving water [7], and so forth. On the other hand, directing the behavior of a process with fractional-order controllers would be an advantage, because the responses are not restricted to a sum of exponential functions; therefore, a wide range of responses neglected by integer-order calculus would be approached [8]. For other advantages of fractional calculus, we can see real materials [9–13], control engineering [14, 15], electromagnetism [16], biosciences [17], fluid mechanics [18], electrochemistry [19], diffusion processes [20], dynamic of viscoelastic materials [21], viscoelastic systems [22], continuum and statistical mechanics [23], propagation of spherical flames [24], robotic manipulators [25], gear transmissions [26], and vibration systems [27]. It is well known that the fractional-order differential and integral operators are nonlocal operators. This is one reason why fractional differential operators provide an excellent instrument for description of memory and hereditary properties of various physical processes.

In recent years, there have been some works on the existence of solutions (or mild solutions) for semilinear fractional differential equations, see [28–36]. They use mainly Krasnoselskii’s fixed-point theorem, Leray-Schauder fixed-point theorem, or contraction mapping principle. They established various criteria on the existence and uniqueness of solutions (or mild solutions) for the semilinear fractional differential equations by considering an integral equation which is given in terms of probability density functions and operator semigroups. Many partial differential equations involving time-variable t can turn to semilinear fractional differential equations in Banach spaces; they always generate an unbounded closed operator term A, such as the time fractional diffusion equation of order \( \alpha \in (0,1) \), namely,

\[
\partial_\tau^\alpha u(y, t) = Au(y, t), \quad t \geq 0, \; y \in \mathbb{R},
\]

where A may be linear fractional partial differential operator. So, (1.1) has the extensive application value.

However, to the authors’ knowledge, no studies considered the periodic boundary value problems for the abstract semilinear fractional differential equations involving the operator semigroup generator \(-A\). Our results can be considered as a contribution to this emerging field. We use the method of upper and lower solutions coupled with monotone iterative technique and the characteristics of positive operators semigroup.

The method of upper and lower solutions has been effectively used for proving the existence results for a wide variety of nonlinear problems. When coupled with monotone iterative technique, one obtains the solutions of the nonlinear problems besides enabling the study of the qualitative properties of the solutions. The basic idea of this method is that using the upper and lower solutions as an initial iteration, one can construct monotone sequences, and these sequences converge monotonically to the maximal and minimal solutions. In some papers, some existence results for minimal and maximal solutions are obtained by establishing comparison principles and using the method of upper and lower solutions and
the monotone iterative technique. The method requires establishing comparison theorems which play an important role in the proof of existence of minimal and maximal solutions. In abstract semilinear fractional differential equations, positive operators semigroup can play this role, see Li [37–41].

In Section 2, we introduce some useful preliminaries. In Section 3, in two cases: $T(t)$ is compact or noncompact, we establish various criteria on existence of the minimal and maximal mild solutions of PBVP (1.1). The method of upper and lower solutions coupled with monotone iterative technique, and the characteristics of positive operators semigroup are applied effectively. In Section 4, we give also an example to illustrate the applications of the abstract results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

If $-A$ is the infinitesimal generator of a $C_0$-semigroup in a Banach space, then $-(A + qI)$ generates a uniformly bounded $C_0$-semigroup for $q > 0$ large enough. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of a $C_0$-semigroup to the case in which the semigroup is uniformly bounded. Hence, for convenience, throughout this paper, we suppose that $-A$ is the infinitesimal generator of a uniformly bounded $C_0$-semigroup $\{T(t)\}_{t \geq 0}$. This means that there exists $M \geq 1$ such that

$$\|T(t)\| \leq M, \quad t \geq 0. \quad (2.1)$$

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1 (see [9, 32]). The fractional integral of order $\alpha$ with the lower limit zero for a function $f \in AC[0, \infty)$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \quad 0 < \alpha < 1, \quad (2.2)$$

provided the right side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (see [9, 32]). The Riemann-Liouville derivative of order $\alpha$ with the lower limit zero for a function $f \in AC[0, \infty)$ can be written as

$$D^\alpha f(t) = \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \quad 0 < \alpha < 1. \quad (2.3)$$

Definition 2.3 (see [9, 32]). The Caputo fractional derivative of order $\alpha$ for a function $f \in AC[0, \infty)$ can be written as

$$D^\alpha f(t) = \frac{d}{dt} I^{1-\alpha} (f(t) - f(0)), \quad t > 0, \quad 0 < \alpha < 1. \quad (2.4)$$
Remark 2.4 (see [32]). (i) If $f \in C^1[0, \infty)$, then

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(s)(t-s)^{-\alpha}ds, \quad t > 0, \ 0 < \alpha < 1.$$  \hfill (2.5)

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If $f$ is an abstract function with values in $X$, then the integrals and derivatives which appear in Definitions 2.1–2.3 are taken in Bochner’s sense.

For more fractional theories, one can refer to the books [9, 42–44].

Throughout this paper, let $X$ be an ordered Banach space with norm $\| \cdot \|$ and partial order $\leq$, whose positive cone $P = \{ y \in X \mid y \geq \theta \}$ ($\theta$ is the zero element of $X$) is normal with normal constant $N$. $X_1$ denotes the Banach space $D(A)$ with the graph norm $\| \cdot \| = \| \cdot \| + \| A \cdot \|$. Let $C(I, X)$ be the Banach space of all continuous $X$-value functions on interval $I$ with norm $\| u \|_C = \max_{t \in I} \| u(t) \|$. For $u, v \in C(I, X)$, $u \leq v$ if $u(t) \leq v(t)$ for all $t \in I$. For $v, w \in C(I, X)$, denote the ordered interval $[v, w] = \{ u \in C(I, X) \mid v \leq u \leq w \}$ and $[v(t), w(t)] = \{ y \in X \mid v(t) \leq y \leq w(t) \}$, $t \in I$. Set $C^\alpha(I, X) = \{ u \in C(I, X) \mid D^\alpha u \text{ exists and } D^\alpha u \in C(I, X) \}$.

Definition 2.5. If $v_0 \in C^\alpha(I, X) \cap C(I, X_1)$ and satisfies

$$D^\alpha v_0(t) + Av_0(t) \leq f(t, v_0(t)), \quad t \in I,$$

$$v_0(0) \leq v(\omega),$$

then $v_0$ is called a lower solution of PBVP (1.1); if all inequalities of (2.6) are inverse, one calls it an upper solution of PBVP (1.1).

Definition 2.6 (see [29, 45]). If $h \in C(I, X)$, by the mild solution of LIVP,

$$D^\alpha u(t) + Au(t) = h(t), \quad t \in I,$$

$$u(0) = x_0 \in X,$$

one means that the function $u \in C(I, X)$ and satisfies

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{\alpha-1}V(t-s)h(s)ds,$$  \hfill (2.8)

where

$$U(t) = \int_0^\infty \zeta_\alpha(\theta)T(t^\alpha \theta) d\theta,$$

$$V(t) = \alpha \int_0^\infty \theta^{\alpha-1} \zeta_\alpha(\theta)T(t^\alpha \theta) d\theta,$$  \hfill (2.9)

$$\zeta_\alpha(\theta) = \frac{1}{\alpha} \theta^{1-1/\alpha} \rho_\alpha(\theta^{-1/\alpha}),$$

$$\rho_\alpha(\theta) = \frac{1}{\pi} \sum_{n=0}^\infty (-1)^n \theta^{-an-1} \frac{\Gamma(na+1)}{n!} \sin(n\pi \alpha), \quad \theta \in (0, \infty),$$  \hfill (10.10)

and $\zeta_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$. 

Remark 2.7. (i) [29–31] \( \zeta_\alpha(\theta) \geq 0, \theta \in (0, \infty), \int_0^\infty \zeta_\alpha(\theta)d\theta = 1, \) and \( \int_0^\infty \theta^\alpha \zeta_\alpha(\theta)d\theta = 1/\Gamma(1 + \alpha). \)

(ii) [33, 34, 46, 47] The Laplace transform of \( \zeta_\alpha \) is given by

\[
\int_0^\infty e^{-\theta} \zeta_\alpha(\theta)d\theta = \sum_{n=0}^{\infty} \frac{(-p)^n}{\Gamma(1 + n\alpha)} = E_\alpha(-p),
\]

where \( E_\alpha(\cdot) \) is Mittag-Leffler function (see [42]).

(iii) [48] For \( p < 0, 0 < E_\alpha(p) < E_\alpha(0) = 1. \)

Lemma 2.8. If \( |T(t)|_{t \geq 0} \) is an exponentially stable \( C_0 \)-semigroup, there are constants \( N \geq 1 \) and \( \delta > 0 \), such that

\[
\|T(t)\| \leq Ne^{-\delta t}, \quad t \geq 0,
\]

then the linear periodic boundary value problem (LPBVP)

\[
D^\alpha u(t) + Au(t) = h(t), \quad t \in I,
\]

\[
u(0) = u(\omega)
\]

has a unique mild solution

\[
(Ph)(t) = U(t)B(h) + \int_0^t (t-s)^{\alpha-1}V(t-s)h(s)ds,
\]

where \( U(t) \) and \( V(t) \) are given by (2.9)

\[
B(h) = (I - U(\omega))^{-1} \int_0^\omega (\omega-s)^{\alpha-1}V(\omega-s)h(s)ds.
\]

Proof. In \( X \), give equivalent norm \( \| \cdot \| \) by

\[
|x| = \sup_{t \geq 0} \|e^{\delta t}T(t)x\|,
\]

then \( |x| \leq |x| \leq N\|x\| \). By \( |T(t)| \), we denote the norm of \( T(t) \) in \( (X, \| \cdot \|) \), then for \( t \geq 0 \),

\[
|T(t)x| = \sup_{s \geq 0} \|e^{\delta s}T(s)T(t)x\|
\]

\[
= e^{-\delta t} \sup_{s \geq 0} \|e^{\delta(s+t)}T(s+t)x\|
\]

\[
= e^{-\delta t} \sup_{\eta \geq 0} \|e^{\delta \eta}T(\eta)x\|
\]

\[
\leq e^{-\delta t}|x|.
\]
Thus, $|T(t)| \leq e^{-\sigma t}$. Then by Remark 2.7,

$$|U(\omega)| = \left| \int_0^\infty \zeta_\alpha(\theta) T(\omega^\alpha \theta) d\theta \right|$$

$$\leq \int_0^\infty \zeta_\alpha(\theta) e^{-\delta \omega^\alpha \theta} d\theta$$

$$= E(x)(-\delta \omega^\alpha) < 1. \quad (2.18)$$

Therefore, $I - U(\omega)$ has bounded inverse operator and

$$(I - U(\omega))^{-1} = \sum_{n=0}^{\infty} (U(\omega))^n. \quad (2.19)$$

Set

$$x_0 = (I - U(\omega))^{-1} \int_0^\omega (\omega - s)^{\alpha - 1} V(\omega - s)h(s) ds,$$  

(2.20)

then

$$u(t) = U(t)x_0 + \int_0^t (t - s)^{\alpha - 1} V(t - s)h(s) ds \quad (2.21)$$

is the unique mild solution of LIVP (2.7) and satisfies $u(0) = u(\omega)$. So set

$$B(h) = (I - U(\omega))^{-1} \int_0^\omega (\omega - s)^{\alpha - 1} V(\omega - s)h(s) ds,$$

$$Ph(t) = U(t)B(h) + \int_0^t (t - s)^{\alpha - 1} V(t - s)h(s) ds,$$  

(2.22)

then $Ph$ is the unique mild solution of LPBVP (2.13). \hfill \Box

Remark 2.9. For sufficient conditions of exponentially stable $C_0$-semigroup, one can see [49].

Definition 2.10. A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ is called a compact semigroup if $T(t)$ is compact for $t > 0$.

Definition 2.11. A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ is called an equicontinuous semigroup if $T(t)$ is continuous in the uniform operator topology (i.e., uniformly continuous) for $t > 0$.

Remark 2.12. Compact semigroups, differential semigroups, and analytic semigroups are equicontinuous semigroups, see [50]. In the applications of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroups are analytic semigroups.

Definition 2.13. A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ is called a positive semigroup if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$. 
Lemma 2.16. The operators $U$ and $V$ given by (2.9) have the following properties:

(i) For any fixed $t \geq 0$, $U(t)$ and $V(t)$ are linear and bounded operators, that is, for any $x \in X$, 
\[ \|U(t)x\| \leq M\|x\|, \quad \|V(t)x\| \leq \frac{\alpha M}{1 + t}\|x\|, \quad (2.23) \]

(ii) $\{U(t)\}_{t \geq 0}$ and $\{V(t)\}_{t \geq 0}$ are strongly continuous,
(iii) $\{U(t)\}_{t \geq 0}$ and $\{V(t)\}_{t \geq 0}$ are compact operators if $\{T(t)\}_{t \geq 0}$ is a compact semigroup,
(iv) $U(t)$ and $V(t)$ are continuous in the uniform operator topology (i.e., uniformly continuous) for $t > 0$ if $\{T(t)\}_{t \geq 0}$ is an equicontinuous semigroup,
(v) $U(t)$ and $V(t)$ are positive for $t \geq 0$ if $\{T(t)\}_{t \geq 0}$ is a positive semigroup,
(vi) $(I - U(\omega))^{-1}$ is a positive operator if $\{T(t)\}_{t \geq 0}$ is an exponentially and positive semigroup.

Proof. For the proof of (i)–(iii), one can refer to [29, 31]. We only check (iv), (v), and (vi) as follows.

(iv) For $0 < t_1 \leq t_2$, we have 
\[
\|U(t_2) - U(t_1)\| \leq \int_0^\infty \xi_{\alpha}(\theta) \|T(t_2^\alpha\theta) - T(t_1^\alpha\theta)\| d\theta, \\
\|V(t_2) - V(t_1)\| \leq \alpha \int_0^\infty \theta \xi_{\alpha}(\theta) \|T(t_2^\alpha\theta) - T(t_1^\alpha\theta)\| d\theta. \quad (2.24)
\]

Since $T(t)$ is continuous in the uniform operator topology for $t > 0$, by Lebesgue-dominated convergence theorem and Remark 2.7 (i), $U(t)$ and $V(t)$ are continuous in the uniform operator topology for $t > 0$.

(v) By Remark 2.7 (i), the proof is then complete.

(vi) By (v), (2.18), and (2.19), the proof is then complete.

\[ \square \]

3. Main Results

Case 1. $\{T(t)\}_{t \geq 0}$ is compact.

Theorem 3.1. Assume that $\{T(t)\}_{t \geq 0}$ is a compact and positive semigroup in $X$, PBVP (1.1) has a lower solution $\nu_0$ and an upper solution $\omega_0$ with $\nu_0 \leq \omega_0$ and satisfies the following.
(H) There exists a constant $C > 0$ such that

$$f(t, x_2) - f(t, x_1) \geq -C(x_2 - x_1),$$

for any $t \in I$, and $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$, that is, $f(t, x) + Cx$ is increasing in $x$ for $x \in [v_0(t), w_0(t)]$.

Then PBVP (1.1) has the minimal and maximal mild solutions between $v_0$ and $w_0$, which can be obtained by a monotone iterative procedure starting from $v_0$ and $w_0$, respectively.

**Proof.** It is easy to see that $-(A + CI)$ generates an exponentially stable and positive compact semigroup $S(t) = e^{-Ct}T(t)$. By (2.1), $\|S(t)\| \leq M$. Let $\Phi(t) = \int_0^\infty \xi_s(\theta)S(t^s\theta)\,d\theta, \Psi(t) = \alpha \int_0^\infty \theta \xi_s(\theta)S(t^s\theta)\,d\theta$. By Remark 2.7 (i), we have that

$$\|\Phi(t)\| \leq M, \quad \|\Psi(t)\| \leq \frac{\alpha}{\Gamma(1 + \alpha)} M, \quad t \geq 0.$$ (3.2)

From Lemma 2.8, $(I - \Phi(\omega))$ has bounded inverse operator and

$$(I - \Phi(\omega))^{-1} = \sum_{n=0}^\infty (\Phi(\omega))^n.$$ (3.3)

By Lemma 2.16 (v) and (vi), $\Phi(t)$ and $\Psi(t)$ are positive for $t \geq 0$, and $(I - \Phi(\omega))^{-1}$ is positive. Let $D = [v_0, w_0]$, then we define a mapping $Q : D \to C(I, X)$ by

$$Qu(t) = \Phi(t)B_1(u) + \int_0^t (t - s)^{\alpha - 1}\Psi(t - s)[f(s, u(s)) + Cu(s)]\,ds, \quad t \in I,$$ (3.4)

where

$$B_1(u) = (I - \Phi(\omega))^{-1} \int_0^\infty (\omega - s)^{\alpha - 1}\Psi(\omega - s)[f(s, u(s)) + Cu(s)]\,ds.$$ (3.5)

By the continuity of $f$ and Lemma 2.16 (ii), $Q : D \to C(I, X)$ is continuous. By Lemma 2.8, $u \in D$ is a mild solution of PBVP (1.1) if and only if

$$u = Qu.$$ (3.6)

For $u_1, u_2 \in D$ and $u_1 \leq u_2$, from (H), the positivity of operators $(I - \Phi(\omega))^{-1}, \Phi(t)$, and $\Psi(t)$, we have that

$$Qu_1 \leq Qu_2.$$ (3.7)
Now, we show that \( v_0 \leq Qv_0 \), \( Qw_0 \leq w_0 \). Let \( D^\alpha v_0(t) + Av_0(t) + Cv_0(t) \triangleq \alpha(t) \), by Definition 2.5, the positivity of operator \( \Psi(t) \), we have that

\[
v_0(t) = \Phi(t)v_0(0) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\alpha(s)ds \leq \Phi(t)v_0(0) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[ f(s,v_0(s)) + Cv_0(s) \right]ds, \quad t \in I.
\]

In particular,

\[
v_0(\omega) \leq \Phi(\omega)v_0(0) + \int_0^{\omega} (\omega-s)^{\alpha-1}\Psi(\omega-s)\left[ f(s,v_0(s)) + Cv_0(s) \right]ds.
\]

By Definition 2.5, \( v_0(0) \leq v(\omega) \), and by the positivity of operator \( (I - \Phi(\omega))^{-1} \), we have that

\[
v_0(0) \leq (I - \Phi(\omega))^{-1} \int_0^{\omega} (\omega-s)^{\alpha-1}\Psi(\omega-s)\left[ f(s,v_0(s)) + Cv_0(s) \right]ds = B_1(v_0).
\]

Then by (3.8) and the positivity of operator \( \Phi(t) \),

\[
v_0(t) \leq \Phi(t)B_1(v_0) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[ f(s,v_0(s)) + Cv_0(s) \right]ds = (Qv_0)(t), \quad t \in I,
\]

namely, \( v_0 \leq Qv_0 \). Similarly, we can show that \( Qw_0 \leq w_0 \). For \( u \in D \), in view of (3.7), then \( v_0 \leq Qv_0 \leq Qu \leq Qw_0 \leq w_0 \). Thus, \( Q : D \rightarrow D \) is a continuous increasing monotonic operator. We can now define the sequences

\[
v_n = Qv_{n-1}, \quad w_n = Qw_{n-1}, \quad n = 1, 2, \ldots,
\]

and it follows from (3.7) that

\[
v_0 \leq v_1 \leq \cdots v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0.
\]

In the following, we prove that \( \{v_n\} \) and \( \{w_n\} \) are convergent in \( C(I,X) \). First, we show that \( QD = \{Qu \mid u \in D\} \) is precompact in \( C(I,X) \). Let

\[
(Wu)(t) = \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[ f(s,u(s)) + Cu(s) \right]ds, \quad t \in I,
\]
then we prove that for all $0 < t \leq \omega$, $(WD)(t) = ((Wu)(t) \mid u \in D)$ is precompact in $X$. For $0 < \varepsilon < t$, let

$$(W\varepsilon u)(t) = \int_0^{t-\varepsilon} (t-s)^{a-1} \Psi(t-s) \left[f(s, u(s)) + Cu(s)\right] ds$$

$$= \int_0^{t-\varepsilon} (t-s)^{a-1} \left(\alpha \int_0^\infty \theta \zeta_s (\theta) S((t-s)^a \theta - \varepsilon) d\theta\right) \left[f(s, u(s)) + Cu(s)\right] ds$$

$$= S(\varepsilon) \int_0^{t-\varepsilon} (t-s)^{a-1} \left(\alpha \int_0^\infty \theta \zeta_s (\theta) S((t-s)^a \theta - \varepsilon) d\theta\right) \left[f(s, u(s)) + Cu(s)\right] ds.$$  \hspace{1cm} (3.15)

For $u \in D$, by (H), $f(t, v_0(t)) + C v_0(t) \leq f(t, u(t)) + Cu(t) \leq f(t, \omega_0(t)) + C \omega_0(t)$ for $0 \leq t \leq \omega$. By the normality of the cone $P$, there is $M_1 > 0$ such that

$$\|f(t, u(t)) + Cu(t)\| \leq M_1, \quad 0 \leq t \leq \omega. \hspace{3cm} (3.16)$$

Thus, by (3.16) and Remark 2.7 (i), we have

$$\left\| \int_0^{t-\varepsilon} (t-s)^{a-1} \left(\alpha \int_0^\infty \theta \zeta_s (\theta) S((t-s)^a \theta - \varepsilon) d\theta\right) \left[f(s, u(s)) + Cu(s)\right] ds \right\|$$

$$\leq M_1 \int_0^{t-\varepsilon} (t-s)^{a-1} \left(\alpha \int_0^\infty \theta \zeta_s (\theta) \|S((t-s)^a \theta - \varepsilon)\| d\theta\right) ds$$

$$\leq MM_1 \int_0^{t-\varepsilon} (t-s)^{a-1} \left(\alpha \int_0^\infty \theta \zeta_s (\theta) d\theta\right) ds$$

$$= MM_1 \frac{\alpha}{\Gamma(1 + \alpha)} \int_0^{t-\varepsilon} (t-s)^{a-1} ds$$

$$= MM_1 \frac{(t^a - \varepsilon^a)}{\Gamma(1 + \alpha)}, \quad 0 < t \leq \omega. \hspace{3cm} (3.17)$$

Then by (3.15), (3.17) and the compactness of $S(\varepsilon)$, for $0 < t \leq \omega$, $(W\varepsilon D)(t) = \{(W\varepsilon u)(t) \mid u \in D\}$ is precompact in $X$. Furthermore, by (3.16) and Lemma 2.16 (i), we have

$$\|(Wu)(t) - (W\varepsilon u)(t)\| = \left\| \int_0^{t} (t-s)^{a-1} \Psi(t-s) \left[f(s, u(s)) + Cu(s)\right] ds \right\|$$

$$\leq MM_1 \frac{\alpha}{\Gamma(1 + \alpha)} \int_{t-\varepsilon}^{t} (t-s)^{a-1} ds$$

$$= MM_1 \frac{e^a}{\Gamma(1 + \alpha)}.$$  \hspace{1cm} (3.18)

Therefore, for $0 < t \leq \omega$, $(WD)(t)$ is precompact in $X$. In particular, $(WD)(\omega)$ is precompact in $X$, and then $B_1(D) = (I - \Phi(\omega))^{-1}(WD)(\omega)$ is precompact. Then in view of Lemma 2.16 (i), $(QD)(t) = \{(Qu)(t) \mid u \in D\} = \Phi(t)B_1(D) + (WD)(t)$ is precompact in $X$ for $0 \leq t \leq \omega$. 


Furthermore, for \(0 \leq t_1 < t_2 \leq \omega\), by (3.16) and Lemma 2.16 (i) we have that

\[
\|(Wu)(t_2) - (Wu)(t_1)\| = \left\| \int_0^{t_1} (t_2 - s)^{a-1}\Psi(t_2 - s)\left[f(s, u(s)) + Cu(s)\right]ds \\
- \int_0^{t_1} (t_1 - s)^{a-1}\Psi(t_1 - s)\left[f(s, u(s)) + Cu(s)\right]ds \right\|
\leq M_1 \int_0^{t_1} \left\| (t_2 - s)^{a-1}\Psi(t_2 - s) - (t_1 - s)^{a-1}\Psi(t_1 - s) \right\| ds \\
+ MM_1 \frac{\alpha}{\Gamma(1 + \alpha)} \int_t^{t_2} (t_2 - s)^{a-1} ds
\leq M_1 \int_0^{t_1} (t_2 - s)^{a-1}\Psi(t_2 - s) - \Psi(t_1 - s) \right\| ds \\
+ MM_1 \frac{\alpha}{\Gamma(1 + \alpha)} \left[ t_1^\alpha - (t_2 - t_1)^\alpha \right] + MM_1 \frac{\alpha}{\Gamma(1 + \alpha)} (t_2 - t_1)^\alpha
\leq M_1(t_2 - t_1)^{a-1} \int_0^{t_1} \left\| \Psi(t_2 - s) - \Psi(t_1 - s) \right\| ds + 2MM_1 \frac{\alpha}{\Gamma(1 + \alpha)} (t_2 - t_1)^\alpha
+ MM_1 \frac{\alpha}{\Gamma(1 + \alpha)} (t_2 - t_1).
\]

By Remark 2.12 and Lemma 2.16 (iv), \(\Psi(t)\) is continuous in the uniform operator topology for \(t > 0\). Then by Lebesgue-dominated convergence theorem, \(WD\) is equicontinuous in \(C(I, X)\). By Lemma 2.16 (ii), \(\{\Psi(t)\}_{t \geq 0}\) is strongly continuous. So, \(QD\) is equicontinuous in \(C(I, X)\).

Then by Ascoli-Arzela’s theorem, \(QD = \{Qu \mid u \in D\}\) is precompact in \(C(I, X)\). By (3.12) and (3.13), \(\{v_n\}\) has a convergent subsequence in \(C(I, X)\). Combining this with the monotonicity of \(\{v_n\}\), it is itself convergent in \(C(I, X)\). Using a similar argument to that for \(\{v_n\}\), we can prove that \(\{w_n\}\) is also convergent in \(C(I, X)\). Set

\[
u = \lim_{n \to \infty} v_n, \quad \overline{u} = \lim_{n \to \infty} w_n.
\]

Let \(n \to \infty\), by the continuity of \(Q\) and (3.12), we have

\[
u = Qu, \quad \overline{u} = Q\overline{u}.
\]

By (3.7), if \(u \in D\) is a fixed-point of \(Q\), then \(v_1 = Qv_0 \leq Qu = u \leq Qu_0 = w_1\). By induction, \(v_n \leq u \leq w_n\). By (3.13) and taking the limit as \(n \to \infty\), we conclude that \(v_0 \leq \nu \leq u \leq \overline{u} \leq w_0\). This means that \(\nu, \overline{u}\) are the minimal and maximal fixed-points of \(Q\) on \([v_0, w_0]\), respectively. By (3.6), they are the minimal and maximal mild solutions of PBVP (1.1) on \([v_0, w_0]\), respectively. \(\square\)
**Theorem 3.2.** Assume that \( \{T(t)\}_{t \geq 0} \) is a compact and positive semigroup in \( X \), \( f(t, \theta) \geq \theta \) for \( t \in I \). If there is \( y \in X \) such that \( y \geq \theta \), \( Ay \geq f(t, y) \) for \( t \in I \), and \( f \) satisfies the following:

\[(H_1) \text{ There exists a constant } C_1 > 0 \text{ such that} \]
\[f(t, x_2) - f(t, x_1) \geq -C_1(x_2 - x_1), \quad (3.22)\]

for any \( t \in I \), and \( \theta \leq x_1 \leq x_2 \leq y \), that is, \( f(t, x) + C_1 x \) is increasing in \( x \) for \( x \in [\theta, y] \).

Then PBVP (1.1) has a positive mild solution \( u: \theta \leq u \leq y \).

**Proof.** Let \( v_0 = \theta \) and \( w_0 = y \), by Theorem 3.1, PBVP (1.1) has mild solution on \([v_0, w_0]\). \( \square \)

Case 2. \( \{T(t)\}_{t \geq 0} \) is noncompact.

**Theorem 3.3.** Assume that the positive cone \( P \) is regular, \( \{T(t)\}_{t \geq 0} \) is an equicontinuous and positive semigroup in \( X \), PBVP (1.1) has a lower solution \( v_0 \) and an upper solution \( w_0 \) with \( v_0 \leq w_0 \), and \( (H) \) holds, then PBVP (1.1) has the minimal and maximal mild solutions between \( v_0 \) and \( w_0 \), which can be obtained by a monotone iterative procedure starting from \( v_0 \) and \( w_0 \), respectively.

**Proof.** By the proof of Theorem 3.1, (3.2)–(3.13) and (3.19) are valid. By Lemma 2.16 (iv), \( \Psi(t) \) is continuous in the uniform operator topology for \( t > 0 \). Then by Lebesgue-dominated convergence theorem, \( WD \) is equicontinuous in \( C(I, X) \). From Lemma 2.16 (ii), \( \{\Psi(t)\}_{t \geq 0} \) is strongly continuous. So, \( QD \) is equicontinuous in \( C(I, X) \). Thus, \( \{Qv_n\} \) is equicontinuous in \( C(I, X) \).

For \( 0 \leq t \leq \omega \), by (3.7) and (3.13), \( \{(Qv_n)(t)\} \) is monotone in \( X \). Since the cone \( P \) is regular, then \( \{(Qv_n)(t)\} \) is convergent in \( X \).

By Ascoli-Arzela’s theorem, \( \{Qv_n\} \) is precompact in \( C(I, X) \) and \( \{Qv_n\} \) has a convergent subsequence in \( C(I, X) \). Combining this with the monotonicity of \( \{Qv_n\} \), it is itself convergent in \( C(I, X) \). Using a similar argument to that for \( \{Qw_n\} \), we can prove that \( \{Qw_n\} \) is also convergent in \( C(I, X) \).

Let
\[\underline{u} = \lim_{n \to \infty} v_n = \lim_{n \to \infty} Qv_{n-1}, \quad \bar{u} = \lim_{n \to \infty} w_n = \lim_{n \to \infty} Qw_{n-1}, \quad (3.23)\]
then it is similar to the proof of Theorem 3.1 that \( \underline{u} \) and \( \bar{u} \) are the minimal and maximal mild solutions of PBVP (1.1) on \([v_0, w_0]\), respectively. \( \square \)

**Corollary 3.4.** Let \( X \) be an ordered and weakly sequentially complete Banach space. Assume that \( \{T(t)\}_{t \geq 0} \) is an equicontinuous and positive semigroup in \( X \), PBVP (1.1) has a lower solution \( v_0 \) and an upper solution \( w_0 \) with \( v_0 \leq w_0 \), and \( (H) \) holds, then PBVP (1.1) has the minimal and maximal mild solutions between \( v_0 \) and \( w_0 \), which can be obtained by a monotone iterative procedure starting from \( v_0 \) and \( w_0 \), respectively.

**Proof.** In an ordered and weakly sequentially complete Banach space, the normal cone \( P \) is regular. Then the proof is complete. \( \square \)

**Corollary 3.5.** Let \( X \) be an ordered and reflective Banach space. Assume that \( \{T(t)\}_{t \geq 0} \) is an equicontinuous and positive semigroup in \( X \), PBVP (1.1) has a lower solution \( v_0 \) and an upper solution \( w_0 \) with \( v_0 \leq w_0 \), and \( (H) \) holds, then PBVP (1.1) has the minimal and maximal mild
solutions between \( v_0 \) and \( w_0 \), which can be obtained by a monotone iterative procedure starting from \( v_0 \) and \( w_0 \), respectively.

**Proof.** In an ordered and reflective Banach space, the normal cone \( P \) is regular. Then the proof is complete. \( \square \)

By Theorem 3.3, Corollaries 3.4 and 3.5, we have the following.

**Corollary 3.6.** Assume that \( \{ T(t) \}_{t \geq 0} \) is an equicontinuous and positive semigroup in \( X \), \( f(t, \theta) \geq \theta \) for \( t \in I \). If there is \( y \in X \) such that \( y \geq \theta \), \( Ay \geq f(t, y) \) for \( t \in I \), \( f \) satisfies (H1) and one of the following conditions:

1. \( X \) is an ordered Banach space, whose positive cone \( P \) is regular,
2. \( X \) is an ordered and weakly sequentially complete Banach space,
3. \( X \) is an ordered and reflective Banach space.

then PBVP (1.1) has positive mild solution \( u : \theta \leq u \leq y \).

4. **Examples**

**Example 4.1.** Consider the following periodic boundary value problem for fractional parabolic partial differential equations in \( X \):

\[
\partial_t^\alpha u + A(x, D)u = g(x, t, u), \quad (x, t) \in \Omega \times I,
Bu = 0, \quad (x, t) \in \partial \Omega \times I,
u(x, 0) = u(x, \omega), \quad x \in \Omega,
\]

where \( \partial_t^\alpha \) is the Caputo fractional partial derivative with order \( 0 < \alpha < 1 \), \( I = [0, \omega] \), \( \Omega \subset \mathbb{R}^N \) is a bounded domain with a sufficiently smooth boundary \( \partial \Omega \), \( g : \overline{\Omega} \times I \times \mathbb{R} \to \mathbb{R} \) is continuous, \( Bu = b(x)u + \delta(\partial u / \partial n) \) is a regular boundary operator on \( \partial \Omega \), and

\[
A(x, D)u = -\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial y_j} \right)
\]

is a symmetrical strong elliptic operator of second order, whose coefficient functions are Hölder continuous in \( \Omega \).

Let \( X = L^p(\Omega) (p \geq 2), P = \{ v \mid v \in L^p(\Omega), v(x) \geq 0 \text{ a.e. } x \in \Omega \} \), then \( X \) is a Banach space, and \( P \) is a regular cone in \( X \). Define the operator \( A \) as follows:

\[
D(A) = \left\{ u \in W^{2,p}(\Omega) \mid Bu = 0 \right\}, \quad Au = A(x, D)u.
\]

Then \( -A \) generates a uniformly bounded analytic semigroup \( T(t) (t \geq 0) \) in \( X \) (see [39]). By the maximum principle, we can easily find that \( T(t) (t \geq 0) \) is positive (see [39]). Let
Mathematical Problems in Engineering

\[ u(t) = u(\cdot, t), \quad f(t, u) = g(\cdot, t, u(\cdot, t)), \] then the problem (4.1) can be transformed into the following problem:

\[
D^\alpha u(t) + Au(t) = f(t, u(t)), \quad t \in I, \\
u(0) = u(\omega).
\]  

**Theorem 4.2.** Let \( f(x, t, 0) \geq 0. \) If there exists \( \omega_0(x, t) \in C^{2,\alpha} (\Omega \times I) \) such that

\[
\partial^\alpha_t \omega_0 + A(x, D)\omega_0 \geq g(x, t, \omega_0), \quad (x, t) \in \Omega \times I, \\
B\omega = 0, \quad (x, t) \in \partial\Omega \times I, \\
\omega_0(x, 0) \geq \omega_0(x, \omega), \quad x \in \Omega,
\]

and \( g \) satisfies the following:

(H4) there exists a constant \( C_2 \geq 0 \) such that

\[
g(x, t, \xi_2) - g(x, t, \xi_1) \geq -C_2(\xi_2 - \xi_1),
\]

for any \( t \in I, \) and \( 0 \leq \xi_1 \leq \xi_2 \leq \omega_0. \)

Then PBVP (4.1) has a mild solution \( u : 0 \leq u \leq \omega_0. \)

**Proof.** Set \( \nu_0 = 0, \) by Theorem 3.3, PBVP (4.1) has the minimal and maximal solutions between 0 and \( \omega_0. \)

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References


