ON A PARTITION PROBLEM OF CANFIELD AND WILF

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Abstract  
Let $A$ and $M$ be nonempty sets of positive integers. A partition of the positive integer $n$ with parts in $A$ and multiplicities in $M$ is a representation of $n$ in the form $n = \sum_{a \in A} m_a a$ where $m_a \in M \cup \{0\}$ for all $a \in A$, and $m_a \in M$ for only finitely many $a$. Denote by $p_{A,M}(n)$ the number of partitions of $n$ with parts in $A$ and multiplicities in $M$. It is proved that there exist infinite sets $A$ and $M$ of positive integers whose partition function $p_{A,M}$ has weakly superpolynomial but not superpolynomial growth. The counting function of the set $A$ is $A(x) = \sum_{a \in A, a \leq x} 1$. It is also proved that $p_{A,M}$ must have at least weakly superpolynomial growth if $M$ is infinite and $A(x) \gg \log x$.

—To the memory of John Selfridge

1. Partition Problems With Restricted Multiplicities

Let $N$ denote the set of positive integers and let $A$ be a nonempty subset of $N$. A partition of $n$ with parts in $A$ is a representation of $n$ in the form

$$n = \sum_{a \in A} m_a a$$

where $m_a \in N \cup \{0\}$ for all $a \in A$, and $m_a \in N$ for only finitely many $a$. The partition function $p_A(n)$ counts the number of partitions of $n$ with parts in $A$. If $\gcd(A) = d > 1$, then $p_A(n) = 0$ for all $n$ not divisible by $d$, and so $p_A(n) = 0$ for infinitely many positive integers $n$. If $p_A(n) \geq 1$ for all sufficiently large $n$, then $\gcd(A) = 1$.

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If $A = \{a_1, \ldots, a_k\}$ is a set of $k$ relatively prime positive integers, then Schur [8] proved that
\[ p_A(n) \sim \frac{n^{k-1}}{(k-1)!a_1a_2\cdots a_k}. \]  
(1)

Nathanson [6] gave a simpler proof of the more precise result:
\[ p_A(n) = \frac{n^{k-1}}{(k-1)!a_1a_2\cdots a_k} + O\left(n^{k-2}\right). \]  
(2)

An arithmetic function is a real-valued function whose domain is the set of positive integers. An arithmetic function $f$ has polynomial growth if there is a positive integer $k$ and an integer $N_0(k)$ such that $1 \leq f(n) \leq n^k$ for all $n \geq N_0(k)$. Equivalently, $f$ has polynomial growth if
\[ \limsup_{n \to \infty} \frac{\log f(n)}{\log n} < \infty. \]

We shall call an arithmetic function nonpolynomial or weakly superpolynomial if it does not have polynomial growth. Thus, the function $f$ is weakly superpolynomial if for every positive integer $k$ there are infinitely many positive integers $n$ such that $f(n) > n^k$, or, equivalently, if
\[ \limsup_{n \to \infty} \frac{\log f(n)}{\log n} = \infty. \]

An arithmetic function $f$ has superpolynomial growth if for every positive integer $k$ we have $f(n) > n^k$ for all sufficiently large integers $n$. Equivalently,
\[ \lim_{n \to \infty} \frac{\log f(n)}{\log n} = \infty. \]

In the following section we construct strictly increasing arithmetic functions that are weakly superpolynomial but not superpolynomial.

The asymptotic formula (1) implies the following result of Nathanson [5, Theorem 15.2, pp. 458–461].

**Theorem 1.** If $A$ is an infinite set of integers and $\gcd(A) = 1$, then $p_A(n)$ has superpolynomial growth.

Canfield and Wilf [2] studied the following variation of the classical partition problem. Let $A$ and $M$ be nonempty sets of positive integers. A partition of $n$ with parts in $A$ and multiplicities in $M$ is a representation of $n$ in the form
\[ n = \sum_{a \in A} m_a a \]
where \( m_a \in M \cup \{0\} \) for all \( a \in A \), and \( m_a \in M \) for only finitely many \( a \). The associated partition function \( p_{A,M}(n) \) counts the number of partitions of \( n \) with parts in \( A \) and multiplicities in \( M \). Note that \( p_{A,M}(0) = 1 \) and \( p_{A,M}(n) = 0 \) for all \( n < 0 \).

Let \( A \) and \( M \) be infinite sets of positive integers such that \( p_{A,M}(n) \geq 1 \) for all sufficiently large \( n \). Canfield and Wilf (“Unsolved problem 1” in [2]) asked if the partition function \( p_{A,M}(N) \) must have weakly superpolynomial growth. The question can be rephrased as follows: Do there exist infinite sets \( A \) and \( B \) of positive integers such that \( p_{A,M}(n) \geq 1 \) for all sufficiently large \( n \) and the partition function \( p_{A,M}(N) \) has polynomial growth? This beautiful problem is still unsolved.

The goal of this paper is to construct infinite sets \( A \) and \( M \) of positive integers such that the partition function \( p_{A,M}(N) \) is weakly superpolynomial but not superpolynomial.

2. Weakly Superpolynomial Functions

Polynomial and superpolynomial growth functions were first studied in connection with the growth of finitely and infinitely generated groups (cf. Milnor [4], Grigorchuk and Pak [3], Nathanson [7]). Growth functions of infinite groups are always strictly increasing, but even strictly increasing functions that do not have polynomial growth are not necessarily superpolynomial.

We note that an arithmetic function \( f \) is weakly superpolynomial but not superpolynomial if and only if

\[
\limsup_{n \to \infty} \frac{\log f(n)}{\log n} = \infty
\]

and

\[
\liminf_{n \to \infty} \frac{\log f(n)}{\log n} < \infty.
\]

In this section we construct a strictly increasing arithmetic function that is weakly superpolynomial but not polynomial.

Let \( (n_k)_{k=1}^\infty \) be a sequence of positive integers such that \( n_1 = 1 \) and

\[
n_{k+1} > 2n_k^k
\]

for all \( k \geq 1 \). We define the arithmetic function

\[
f(n) = n_k^k + (n - n_k) \quad \text{for } n_k \leq n < n_{k+1}.
\]

This function is strictly increasing because

\[
n_k^k - n_k \leq n_{k+1}^{k+1} - n_{k+1}.
\]
for all $k \geq 1$. We have
\[
\lim_{k \to \infty} \frac{\log f(n_k)}{\log n_k} = \lim_{k \to \infty} \frac{k \log n_k}{\log n_k} = \infty
\]
and so
\[
\limsup_{n \to \infty} \frac{\log f(n)}{\log n} = \infty.
\]
Therefore, the function $f$ does not have polynomial growth.

For every positive integer $n$ there is a positive integer $k$ such that $n_k \leq n < n_{k+1}$. Then $f(n) = n + n_k^k - n_k \geq n$ and so
\[
\liminf_{n \to \infty} \frac{\log f(n)}{\log n} \geq 1. \tag{3}
\]
The inequalities
\[
f(n_{k+1} - 1) = n_k^k + (n_{k+1} - 1 - n_k) < \frac{3n_{k+1}}{2}
\]
and
\[
n_{k+1} - 1 > \frac{n_{k+1}}{2}
\]
imply that
\[
1 < \frac{\log f(n_{k+1} - 1)}{\log(n_{k+1} - 1)} < \frac{\log(3n_{k+1}/2)}{\log(n_{k+1}/2)} = 1 + \frac{\log 3}{\log(n_{k+1}/2)}
\]
and so
\[
\lim_{k \to \infty} \frac{\log f(n_{k+1} - 1)}{\log(n_{k+1} - 1)} = 1.
\]
Therefore,
\[
\liminf_{n \to \infty} \frac{\log f(n)}{\log n} \leq 1. \tag{4}
\]
Combining (3) and (4), we obtain
\[
\liminf_{n \to \infty} \frac{\log f(n)}{\log n} = 1.
\]
Thus, the function $f$ has weakly superpolynomial but not superpolynomial growth.

3. Weakly Superpolynomial Partition Functions

**Theorem 2.** Let $a$ be an integer, $a \geq 2$, and let $A = \{a^i\}_{i=0}^\infty$. Let $M$ be an infinite set of positive integers such that $M$ contains $\{1, 2, \ldots, a-1\}$ and no element of $M$ is divisible by $a$. Then $p_{A,M}(n) \geq 1$ for all $n \in \mathbb{N}$, and $p_{A,M}(n) = 1$ for all $n \in A$. In particular, the partition function $p_{A,M}$ does not have superpolynomial growth.
Proof. Every positive integer $n$ has a unique $a$-adic representation, and so $p_{A,M}(n) \geq 1$ for all $n \in \mathbb{N}$.

We shall prove that, for every positive integer $r$, the only partition of $a^r$ with parts in $A$ and multiplicities in $M$ is $a^r = 1^a a^r$. If there were another representation, then it could be written in the form

$$a^r = \sum_{i=1}^{k} m_i a^{j_i},$$

where $k \geq 2$, $m_i \in M$ for $i = 1, \ldots, k$, and $0 \leq j_1 < j_2 < \cdots < j_k < r$. Then

$$a^{r-j_1} = m_1 + a \sum_{i=2}^{k} m_i a^{j_i-j_1-1}.$$

We have $j_i - j_1 - 1 \geq 0$ for $i = 2, \ldots, k$, and so $m_1$ is divisible by $a$, which is absurd. Therefore, $p_{A,M}(a^r) = 1$ for all $r \geq 0$. It follows that

$$\liminf_{n \to \infty} \frac{\log p_{A,M}(n)}{\log n} = \liminf_{r \to \infty} \frac{\log p_{A,M}(a^r)}{\log a^r} = 0$$

and so the partition function $p_{A,M}$ is not superpolynomial. \qed

**Theorem 3.** Let $A$ and $M$ be infinite sets of positive integers. If $A(x) \geq c \log x$ for some $c > 0$ and all $x \geq x_0(A)$, then for every positive integer $k$ there exist infinitely many integers $n$ such that

$$p_{A,M}(n) > n^k.$$

In particular, the partition function $p_{A,M}$ is weakly superpolynomial.

**Proof.** Let $x \geq 1$ and let

$$A(x) = \sum_{\substack{a \in A \leq x}} 1 \quad \text{and} \quad M(x) = \sum_{\substack{m \in M \leq x}} 1$$

denote the counting functions of the sets $A$ and $M$, respectively. If $n \leq x$ and $n = \sum_{a \in A} m_a a$ is a partition of $n$ with parts in $A$ and multiplicities in $M \cup \{0\}$, then $a \leq x$ and $m_a \leq x$, and so

$$\max \{p_{A,M}(n) : n \leq x\} \leq \sum_{n \leq x} p_{A,M}(n) \leq (M(x) + 1)^{A(x)}. \quad (5)$$

Conversely, if the integer $n$ can be represented in the form $n = \sum_{a \in A} m_a a$ with $a \leq x$ and $m_a \leq x$, then $n \leq x^2 A(x) \leq x^3$ and so

$$\sum_{n \leq x^3 A(x)} p_{A,M}(n) \geq (M(x) + 1)^{A(x)} > M(x)^{A(x)}$$
Choose an integer \( n_x \) such that \( n_x \leq x^2 A(x) \) and
\[
p_{A,M}(n_x) = \max \{ p_{A,M}(n) : n \leq x^2 A(x) \}.
\]

Inequality (5) implies that
\[
p_{A,M}(n_x) \leq (M(x^2 A(x)) + 1)^{A(x^2 A(x))}.
\]

Moreover,
\[
M(x)^{A(x)} < \sum_{n \leq x^2 A(x)} p_{A,M}(n) \leq (x^2 A(x) + 1) p_{A,M}(n_x) \leq 2x^3 p_{A,M}(n_x).
\]

It follows that for all \( x \geq x_0(A) \) we have
\[
p_{A,M}(n_x) > \frac{M(x)^{A(x)}}{2x^3} \geq \frac{M(x)^{c \log x}}{2x^3}.
\]

Let \( k \) be a positive integer. Because the set \( M \) is infinite, there exists \( x_1(A,k) \geq x_0(A) \) such that, for all \( x \geq x_1(A,k) \), we have
\[
\log M(x) > \frac{\log 2}{c \log x} + \frac{3k + 3}{c}
\]
and so
\[
p_{A,M}(n_x) > x^{3k} \geq n_x^k.
\]

We shall iterate this process to construct inductively an infinite sequence of pairwise distinct positive integers \( (n_{x_i})_{i=1}^{\infty} \) such that
\[
p_{A,M}(n_{x_i}) > n_{x_i}^k \quad (7)
\]
for all \( i \). Let \( r \geq 1 \), and suppose that a finite sequence of pairwise distinct positive integers \( (n_{x_i})_{i=1}^{r} \) has been constructed such that inequality (7) holds for \( i = 1, \ldots, r \). Choose \( x_{r+1} \) so that
\[
x_{r+1}^{3k} > (M(x_{r}^2 A(x_{r})) + 1)^{A(x_{r}^2 A(x_{r}))}
\]
for all \( i = 1, \ldots, r \), and let \( n_{x_{r+1}} \) be the integer constructed according to procedure above. Applying inequality (6), we obtain
\[
p(n_{x_i}) \leq (M(x_{r}^2 A(x_{r})) + 1)^{A(x_{r}^2 A(x_{r}))}
\]
and so
\[
p(n_{x_{r+1}}) > x_{r+1}^{3k} > p(n_{x_i})
\]
for \( i = 1, \ldots, r \). It follows that \( n_{x_{r+1}} \neq n_{x_i} \) for \( i = 1, \ldots, r \). This completes the induction and the proof. \( \Box \)
Theorem 4. Let \( a \) be an integer, \( a \geq 2 \), and let \( A = \{a^i\}_{i=0}^{\infty} \). Let \( M \) be an infinite set of positive integers such that \( M \) contains \( \{1, 2, \ldots, a - 1\} \) and no element of \( M \) is divisible by \( a \). The partition function \( p_{A,M} \) is weakly superpolynomial but not superpolynomial.

**Proof.** The counting function for the set \( A = \{a^i\}_{i=1}^{\infty} \) is \( A(x) = \lfloor \log x / \log a \rfloor + 1 > \log x / \log a \). By Theorem 3, the partition function \( p_{A,M} \) is weakly superpolynomial. By Theorem 2, the partition function \( p_{A,M} \) is not superpolynomial. This completes the proof. \( \square \)

4. Open Problems

1. We repeat the original problem of Canfield and Wilf: Do there exist infinite sets \( A \) and \( B \) of positive integers such that \( p_{A,M}(n) \geq 1 \) for all sufficiently large \( n \) and the partition function \( p_{A,M}(N) \) has polynomial growth?

2. By Theorem 3, if the partition function \( p_{A,M} \) has polynomial growth, then the set \( A \) must have sub-logarithmic growth, that is, \( A(x) \gg \log x \) is impossible.

   (a) Let \( A = \{k!\}_{k=1}^{\infty} \). Does there exist an infinite set \( M \) of positive integers such that \( p_{A,M}(n) \geq 1 \) for all sufficiently large \( n \) and \( p_{A,M} \) has polynomial growth?

   (b) Let \( A = \{k^k\}_{k=1}^{\infty} \). Does there exist an infinite set \( M \) of positive integers such that \( p_{A,M}(n) \geq 1 \) for all sufficiently large \( n \) and \( p_{A,M} \) has polynomial growth?

3. Let \( A \) be an infinite set of positive integers and let \( M = \mathbb{N} \). Bateman and Erdős [1] proved that the partition function \( p_A = p_{A,N} \) is eventually strictly increasing if and only if \( \gcd(A \setminus \{a\}) = 1 \) for all \( a \in A \). It would be interesting to extend this result to partition functions with restricted multiplicities: Determine a necessary and sufficient condition for infinite sets \( A \) and \( M \) of positive integers to have the property that \( p_{A,M}(n) < p_{A,M}(n+1) \) or \( p_{A,M}(n) \leq p_{A,M}(n+1) \) for all sufficiently large \( n \).

References


