A VARIATION ON PERFECT NUMBERS

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Abstract

For $k \in \mathbb{N}$ we define a new divisor function $s_k$ called the $k^{th}$ prime symmetric function. By analogy with the sum of divisors function $\sigma$, we use the functions $s_k$ to consider variations on perfect numbers, namely $k$-symmetric-perfect numbers as well as $k$-cycles. We find all $k$-symmetric-perfect numbers for $k = 1, 2, 3$. We also consider the problem of whether a natural number $n$ can be expressed in the form $s_k(n)$, and show that for $n$ large enough, it always can be for $k = 1, 2$.

1. Introduction

Definition 1: Let $k$ be a nonnegative integer. We define $s_k : \mathbb{N} \to \mathbb{N} \cup \{0\}$ as follows: If $k = 0$, $s_k(n) \equiv 1$. If $k > 0$, and $n = p_1 \cdots p_r$, where $r = \Omega(n)$ is the number of prime factors (with multiplicity) of $n$, then

$$s_k(n) = \sum p_{i_1} \cdots p_{i_k},$$

where the sum is taken over all products of $k$ prime factors from the set $\{p_1, \ldots, p_r\}$. We say $s_k$ is the $k^{th}$ prime symmetric function.

Note that if $\Omega(n) < k$, we have $s_k(n) = 0$.

There is an alternate way of defining the functions $s_k$. Given $n = p_1 \cdots p_r \in \mathbb{N}$, set

$$S_n(x) = \prod_{i=1}^{r}(x + p_i).$$

Then $s_k(n)$ is the coefficient of $x^{r-k}$ in $S_n(x)$. The empty product is taken to be 1.

Example 1: $s_0(12) = 1$, $s_1(12) = 2 + 2 + 3 = 7$, $s_2(12) = 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 3 = 16$, $s_3(12) = 12$, and $s_4(12) = 0$.

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Several good texts detailing the basic theory of perfect numbers exist, see for instance [1], [4], [7], and [13]. In addition, many variations on perfect numbers have been defined and studied. For examples, see the remaining references. We now define a new variation of perfect, defective, and excessive numbers using the divisor functions $s_k$.

**Definition 2:** Let $n \in \mathbb{N}$. If $s_k(n) < n$, we say $n$ is $k$-symmetric-defective. If $s_k(n) > n$, we say $n$ is $k$-symmetric-excessive. If $s_k(n) = n$, and $\Omega(n) = k$, we say $n$ is trivially $k$-symmetric-perfect. If $s_k(n) = n$, and $\Omega(n) > k$, we say $n$ is $k$-symmetric-perfect. If $n$ is $k$-symmetric-perfect or $k$-symmetric-excessive, we say $n$ is $k$-symmetric-special.

**Notation:** For the sake of brevity we write $k$-SD for $k$-symmetric-defective, $k$-SP for $k$-symmetric-perfect, $k$-SE for $k$-symmetric-excessive, and $k$-SS for $k$-symmetric-special.

**Example 2:** If $p$ is prime, then $p^p$ is a $(p - 1)$-SP number, since

$$s_{p-1}(p^p) = \left( \frac{p}{p-1} \right) p^{p-1} = p^p.$$

In fact, this example has a form of converse:

**Theorem 1:** The prime power $p^\alpha$ is $k$-SP if and only if $\alpha = p$ and $k = p - 1$.

**Proof.** We have seen that this is sufficient, now suppose $k < \alpha$, and $s_k(p^\alpha) = p^\alpha$. Then

$$\left( \frac{\alpha}{k} \right) = p^{\alpha-k}. \quad (1)$$

For $1 < k < \alpha - 1$, $\left( \frac{\alpha}{k} \right)$ is divisible by two distinct prime factors, hence we must have $k = 1$ or $\alpha - 1$. Now $4 = 2^2$, is the only 1-SP number, and corresponds to the case where $k = 1 = \alpha - 1$. Hence we may assume $k = \alpha - 1$, which from (1) implies that $\alpha = p$ and $k = p - 1$. This proves the theorem. \qed

**Definition 3:** A finite sequence $\{n_0, \ldots, n_\ell\}$ is called a $k$-cycle if the following conditions are satisfied:

1. $\ell > 1$,
2. $n_0, \ldots, n_{\ell-1}$ are distinct and $n_\ell = n_0$, and
3. $s_k(n_i) = n_{i+1}$, for $i = 0, 1, \ldots, \ell - 1$.

**2. Basic Properties of Prime Symmetric Functions**

The following proposition is an immediate consequence of the definition.
Proposition 2: If \( n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \), then
\[
s_k(n) = \sum_{i_1 + \cdots + i_r = k} \left( \frac{\alpha_1}{i_1} \right) \cdots \left( \frac{\alpha_r}{i_r} \right) p_1^{i_1} \cdots p_r^{i_r}.
\]

Proposition 3:
\[
s_k(mn) = \sum_{i=0}^{k} s_{k-i}(m)s_i(n).
\]

Proof. If \( m = 1 \), or \( n = 1 \), the result is immediate, as it is if \( k = 0 \). If \( k > 0 \), \( m = p_1 \cdots p_r \), and \( n = q_1 \cdots q_s \), let
\[
S = \{p_1, \ldots, p_r, q_1, \ldots, q_s\}.
\]
Then
\[
s_k(mn) = \sum_{\{r_1, \ldots, r_k\} \subset S} r_1 \cdots r_k.
\]
We collect the terms of this sum having \( k - i \) factors from \( m \), and \( i \) factors from \( n \). The sum of these is equal to \( s_{k-i}(m)s_i(n) \). Summing as \( i \) ranges from 0 to \( k \) gives the desired result.

Corollary 4: Let \( n, k \in \mathbb{N} \), and let \( p \) and \( q \) be primes, with \( p < q \), and suppose \( \Omega(pn) > k \). If \( pn - s_k(pn) > 0 \), then \( pn - s_k(pn) < qn - s_k(qn) \).

Proof. The following inequality
\[
qn - s_k(qn) = qn - qs_{k-1}(n) - s_k(n)
> pn - ps_{k-1}(n) - s_k(n)
= pn - s_k(pn)
\]
is true if \( n > s_{k-1}(n) \). But
\[
qn - s_k(pn) = pn - ps_{k-1}(n) - s_k(n) > 0
\]
by assumption, so
\[
n > s_{k-1}(n) + s_k(n)/p > s_{k-1}(n).
\]

In searching for \( k \)-cycles and \( k \)-SP numbers, it is essential to know when \( s_k(n) \geq n \). We search by fixing \( \Omega(n) \), and systematically checking all products of \( \Omega(n) \) primes. The corollary tells us that if \( s_k(pn) < pn \), then for any \( q > p \), \( qn \) is also \( k \)-SD.
Lemma 5: Let $k, n \in \mathbb{N}$. Then there exists an $r > k$ such that if $\Omega(n) \geq r$, then $n$ is $k$-SD. Let $r(k)$ denote the least such $r$. Then

$$r(k) = \min \{ r : \binom{r}{k} < 2^{r-k} \}.$$ 

Proof. There is an $r > k$ such that the function

$$f(t) = \binom{t}{k}$$

satisfies $f(t) < g(t)$ for all $t \geq r$, where

$$g(t) = 2^{t-k},$$

since $f$ is a polynomial, and $g$ is an exponential function. Now suppose $t \geq r$, and let $p_1, \ldots, p_t$ be the primes. Then

$$\binom{t}{k} = \binom{t}{t-k} < 2^{t-k},$$

which implies

$$\sum \frac{1}{p_{i_1} \cdots p_{i_{t-k}}} \leq \binom{t}{t-k} \frac{1}{2^{t-k}} < 1,$$

where the sum is taken over all $i_1, \ldots, i_{t-k}$ such that $1 \leq i_1 < \cdots < i_{t-k} \leq t$. This implies

$$\sum p_{i_1} \cdots p_{i_k} < p_1 \cdots p_t,$$

where the sum is taken over all $i_1, \ldots, i_k$ such that $1 \leq i_1 < \cdots < i_k \leq t$. This in turn implies that

$$s_k(p_1 \cdots p_t) < p_1 \cdots p_t.$$ 

Now we prove the second statement. The inequality

$$\binom{2k}{k} \geq 2^k$$

holds for all $k \geq 1$, and so $r(k) > 2k$. This in mind, let $r(k)$ be as claimed in the statement of the theorem. We argue inductively. Let $t > r$, and suppose that

$$\binom{t-1}{k} < 2^{t-1-k}.$$ 

Then

$$2\binom{t-1}{k} < 2^{t-k}.$$
Since \( t > 2k \), we have \( t < 2(t - k) \), and so
\[
\binom{t}{k} = \frac{t(t-1) \cdots (t-k+1)}{k!} < \frac{2(t-1)(t-2) \cdots (t-k)}{k!} = 2 \binom{t-1}{k}.
\]
Hence
\[
\binom{t}{k} < 2 \binom{t-1}{k} < 2^{t-k},
\]
and the proof is complete by induction. \( \square \)

The first few values of \( r(k) \) are given in the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( r(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
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<tr>
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<td>9</td>
<td>36</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

The properties of 1-symmetric-perfection etc. corresponding to the first prime symmetric function \( s_1 \) are easily characterized. The primes are the trivial 1-SP numbers, 4 is the only 1-SP number, and all other numbers are 1-SD. Clearly there are no 1-cycles. We now investigate these properties in the second prime symmetric function.

3. The Second Prime Symmetric Function

Let \( n \) be an integer greater than 1. By a family \( E_k(n,r) \) of k-SS numbers, we mean a set
\[
E_k(n,r) = \{ np_1 \cdots p_r | p_1, \ldots, p_r \text{ are primes} \}
\]
such that if \( m \in E_k(n,r) \), then \( m \) is k-SS. The family \( E_2(4,1) \) of numbers of the form \( 4p \), where \( p \) is prime, is one such set, since the elements satisfy \( s_2(4p) = 4p + 4 > 4p \). \( E_k(n,0) \) merely denotes the singleton set of a k-SS number. To find all 2-SP numbers and all 2-cycles we need to find all numbers \( n \) such that \( 2 < \Omega(n) < 6 \), with \( s_2(n) \geq n \), since \( r(2) = 6 \). To do this we use the algorithm mentioned after Corollary 4.
3.1. $\Omega(n) = 3$

\[
\begin{align*}
    s_2(2 \cdot 2 \cdot p) &= 4p + 4 > 4p, \\
    s_2(2 \cdot 3 \cdot p) &= 5p + 6 < 6p, \text{ when } p > 6.
\end{align*}
\]

This shows that there are no other infinite families of 2-SS numbers satisfying $\Omega(n) = 3$. Below we find all 2-SS numbers not belonging to this family.

\[
\begin{align*}
    s_2(2 \cdot 3 \cdot 3) &= 21 > 18, \\
    s_2(2 \cdot 3 \cdot 5) &= 31 > 30, \\
    s_2(2 \cdot 3 \cdot 7) &= 41 < 42, \\
    s_2(3 \cdot 3 \cdot 3) &= 27, \\
    s_2(3 \cdot 3 \cdot 5) &= 39 < 45.
\end{align*}
\]

Thus 27 is the only 2-SP number satisfying $\Omega(n) = 3$. Iterating on the above 2-SE numbers shows none belong to 2-cycles. For example

\[
18 \xrightarrow{s_2} 21 \xrightarrow{s_2} 10 \xrightarrow{s_2} 10 \xrightarrow{s_2} \cdots.
\]

3.2. $\Omega(n) = 4$

\[
\begin{align*}
    s_2(2 \cdot 2 \cdot 2 \cdot p) &= 6p + 12 < 8p, \text{ when } p > 6.
\end{align*}
\]

Thus there are no infinite families of 2-SS numbers with $\Omega(n) = 4$. Iterating on $8p$ for $p = 2, 3, 5$, shows that none belong to a 2-cycle. Checking other cases:

\[
\begin{align*}
    s_2(2 \cdot 2 \cdot 3 \cdot 3) &= 37 > 36, \\
    s_2(2 \cdot 2 \cdot 3 \cdot 5) &= 51 < 60, \\
    s_2(2 \cdot 3 \cdot 3 \cdot 3) &= 45 < 54.
\end{align*}
\]

Hence there are no 2-SP numbers satisfying $\Omega(n) = 4$. Iterating on the above 2-SE numbers shows that none belong to a 2-cycle.

3.3. $\Omega(n) = 5$

\[
\begin{align*}
    s_2(2 \cdot 2 \cdot 2 \cdot 2 \cdot p) &= 8p + 24 < 16p, \text{ when } p > 3.
\end{align*}
\]

Thus there are no infinite families of 2-SS numbers with $\Omega(n) = 5$. Iterating on $16p$ for $p = 2, 3$, shows that 48 is in fact 2-SP, and 32, which is 2-SE, does not belong to a 2-cycle. Checking other cases:

\[
    s_2(2 \cdot 2 \cdot 2 \cdot 3 \cdot 3) = 57 < 72.
\]
Hence 48 is the only 2-SP number satisfying \( \Omega(n) = 5 \). We have proved the following theorem.

**Theorem 6:** 27 and 48 are the only 2-SP numbers.

**Theorem 7:** There are no 2-cycles.

**Proof.** A 2-cycle must have a least element that is 2-SE. We have shown that any such element must belong to the family of numbers of the form \( 4p \). We will show that in all but a few trivial cases \( s_2(s_2(4p)) < 4p \), giving a contradiction. Now, \( s_2(4p) = 8((p + 1)/2) \).

We may assume that \( p \) is odd, and set \( m = (p + 1)/2 \). Thus we will have a contradiction if the following holds:

\[ s_2(8m) < 8m - 4. \]

This is equivalent to

\[ 12 + 6s_1(m) + s_2(m) < 8m - 4, \tag{2} \]

which is equivalent to

\[ \frac{16}{p_1 \ldots p_s} + 6 \sum_{i=1}^{s} \frac{1}{p_1 \ldots \hat{p_i} \ldots p_s} + \sum_{1 \leq i < j \leq s} \frac{1}{p_1 \ldots \hat{p_i} \ldots \hat{p_j} \ldots p_s} < 8, \]

where \( m = p_1 \ldots p_s \). Here \( p_1 \ldots \hat{p_i} \ldots p_s \) is defined to be \( p_1 \ldots p_s/p_i \), and \( p_1 \ldots \hat{p_i} \ldots \hat{p_j} \ldots p_s \) is defined to be \( p_1 \ldots p_s/p_ip_j \).

Since \( p_i \geq 2 \), this expression is implied by:

\[ \frac{16}{2^s} + \frac{6s}{2^{s-1}} + \frac{s(s-1)}{2} \frac{1}{2^{s-2}} < 8, \]

which holds for all \( s \geq 4 \). If \( s = 1 \), then \( m = p \) is prime, and so condition (2) becomes:

\[ 12 + 6p < 8p - 4, \]

which holds for all \( p > 8 \). It is easily verified for \( p = 2, 3, 5 \) and 7, that \( 8p \) does not belong to a 2-cycle.

For \( s = 2 \), we can write \( m = pq \). The only values of \( m \) for which (2) fails are determined by the prime pairs \( (p,q) = (2,2), (2,3) \). In both cases, \( 8m \) does not belong to a 2-cycle.

Finally for \( s = 3 \), if \( m = pqr \), only for the triple \( (p,q,r) = (2,2,2) \) does \( m \) fail to satisfy (2). Again, in this case, \( 8m \) does not belong to a 2-cycle. \( \square \)

**Definition 4:** A sequence \( \{n_i\} \) (finite or infinite) is called a \( k \)-ascending sequence if \( n_i < s_k(n_i) = n_{i+1} \). If \( \{n_i\} = \{n_i\}^t_{i=0} \), then \( \{n_i\} \) is said to have length \( t \).
Remark: The longest 2-ascending sequence is

\[ 8 \rightarrow 12 \rightarrow 16 \rightarrow 24 \rightarrow 30 \rightarrow 31. \]

Definition 5: Let \( k \in \mathbb{N} \cup \{0\} \). We define \( r_k : \mathbb{N} \to \mathbb{N} \cup \{0\} \) by

\[ r_k(n) = |\{s_k^{-1}\{n\}\}| \]

Example 3: \( r_1(1) = 0 \), but for all \( n \geq 2 \), \( r_1(n) \geq 1 \). In fact, \( \lim_{n \to \infty} r_1(n) = \infty \). To see this, simply set

\[ n = s_1(2^a3^b) = 2a + 3b, \]

and observe that the number of pairs \((a, b)\) satisfying this equation can be made arbitrarily large for all \( n \) sufficiently large.

We prove a weaker result for \( r_2 \).

Theorem 8: There exists an \( N \in \mathbb{N} \) such that for all \( m \geq N \), \( r_2(m) \geq 1 \).

Proof. It suffices to show that for \( m \) sufficiently large, \( m = s_2(2^a3^b5^c7^d) \), for some \( a, b, c, \) and \( d \geq 0 \). In general,

\[
s_2(2^a3^b5^c7^d) = \frac{4 \binom{a}{2} + 9 \binom{b}{2} + 25 \binom{c}{2} + 49 \binom{d}{2}}{2} + 6 \binom{a}{1} \binom{b}{1} + 10 \binom{a}{1} \binom{c}{1} + 14 \binom{a}{1} \binom{d}{1} + 15 \binom{b}{1} \binom{c}{1} + 21 \binom{b}{1} \binom{d}{1} + 35 \binom{c}{1} \binom{d}{1}
\]

\[ = \frac{1}{2} \left[ (2a + 3b + 5c + 7d)^2 - (4a + 9b + 25c + 49d) \right] \]

So, given \( m \), we need only find solutions to the equations:

\[
2a + 3b + 5c + 7d = R,
\]

\[
4a + 9b + 25c + 49d = R^2 - 2m,
\]

with nonnegative integers \( a, b, c, d, \) and \( R \in \mathbb{N} \). These equations are equivalent to:

\[
2a - 10c - 28d = 3R - R^2 + 2m, \quad (3)
\]

\[
3b + 15c + 35d = R^2 - 2R - 2m, \quad (4)
\]

Since \( a \) and \( b \) must be nonnegative integers, we have the following necessary and sufficient conditions for a solution to (3) and (4):

1. \( 2m \equiv R^2 + R + d \pmod{3}, \)
2. \(R^2 - 3R - 10c - 28d \leq 2m,\)
3. \(2m \leq R^2 - 2R - 15c - 35d.\)

Note that equation (3) is always satisfied modulo 2. Condition 1 results from taking equation (4) modulo 3, and conditions 2 and 3 are derived from the fact that \(a, b \geq 0.\)

Consider the interval
\[
I_R(c, d) = [R^2 - 3R - 10c - 28d, R^2 - 2R - 15c - 35d].
\]
For fixed \(d,\) let \(c_R(d)\) be the least \(c\) such that \(\ell(I_R(c, d)) < 15,\) where \(\ell(I)\) denotes the length of an interval \(I.\) We use the notation \(L(I)\) and \(R(I)\) to denote the left and right endpoints of an interval \(I,\) respectively. Since \(R(I_R(c, d)) = R(I_R(c + 1, d)) + 15,\) when they exist, we have that
\[
\bigcup_{c=0}^{c_R(d)} I_R(c, d) = [R^2 - 3R - 10c_R(d) - 28d, R^2 - 2R - 35d].
\]
Denote the above interval by \(I_R(d).\) By definition of \(c_R(d),\)
\[
\ell(I_R(c_R(d), d)) = R - 5c_R(d) - 7d < 15, \quad \text{so} \quad -10c_R(d) < -2R + 30 + 14d,
\]
and \(c_R(d)\) is the least such \(c.\) Consider the interval \(\bigcap_{d=0}^{2} I_R(d).\) Clearly \(R(\bigcap_{d=0}^{2} I_R(d)) = R^2 - 2R - 70.\) We now wish to find an upper bound for \(L(\bigcap_{d=0}^{2} I_R(d)).\) From the above inequality, we have that
\[
L(I_R(d)) = R^2 - 3R - 10c_R(d) - 28d < R^2 - 5R - 14d + 30.
\]
Thus
\[
L(\bigcap_{d=0}^{2} I_R(d)) = \max\{R^2 - 3R - 10c_R(d) - 28d | d = 0, 1, 2\}
< \max\{R^2 - 5R - 14d + 30 | d = 0, 1, 2\}
= R^2 - 5R + 30.
\]
Let \(J_R = [R^2 - 5R + 30, R^2 - 2R - 70].\) Then \(J_R \subset \bigcap_{d=0}^{2} I_R(d).\) Now
\[
L(J_{R+1}) \leq R(J_R), \quad \text{if and only if} \quad R^2 - 3R + 26 \leq R^2 - 2R - 70,
\]
which holds for all \(R \geq 96.\) So if \(2m \geq L(J_{96}) = 8766,\) then there is an \(R \geq 96\) such that \(2m \in J_R \subset \bigcap_{d=0}^{2} I_R(d).\) Choose \(d \in \{0, 1, 2\}\) such that condition 1 is satisfied. Since \(2m \in I_R(d),\) there is a \(c \geq 0\) such that \(2m \in I_R(c, d).\) For these values of \(R, c,\) and \(d,\) conditions 2 and 3 are satisfied. In other words, there exists an \(n\) such that \(m = s_2(n).\) This completes the proof.

We end this section with a conjecture.

**Conjecture 1:** For every \(k \in \mathbb{N},\) \(\lim_{n \to \infty} r_k(n) = \infty.\)
4. Higher Prime Symmetric Functions

**Theorem 9:** (1) Let \( n \in \mathbb{N} \). If \( n \) is \( k \)-SS then \( pn \) is \((k+1)\)-SE for every prime \( p \).

(2) If \( pn \) is \((k+1)\)-SS for every prime \( p \), then \( n \) is \( k \)-SS, and hence by (1), \( pn \) is \((k+1)\)-SE for every prime \( p \).

**Proof.** (1) Suppose \( n \) is \( k \)-SS. Then since \( \Omega(n) > k \), we have \( s_{k+1}(n) > 0 \). So

\[
\begin{align*}
s_{k+1}(pn) &= ps_k(n) + s_{k+1}(n) \\
&\geq pn + s_{k+1}(n) \\
&> pn.
\end{align*}
\]

(2) If \( s_{k+1}(pn) = ps_k(n) + s_{k+1}(n) \geq pn \), for every prime \( p \), then \( s_k(n) = n - s_{k+1}(n)/p \). Letting \( p \rightarrow \infty \), we have \( s_k(n) \geq n \). \( \square \)

**Corollary 10:** For \( k \in \mathbb{N} \), there are only finitely many \( k \)-SP numbers.

**Proof.** By the previous theorem, any family \( E_{k+1}(n, r + 1) \) is of the form \( pE_k(n, r) \), where \( p \) ranges over the primes. Furthermore, this family contains only \((k+1)\)-SE numbers. There are only finitely many \((k+1)\)-SE numbers not belonging to any such family. \( \square \)

Thus the infinite families of 3-SE numbers are: \( E_3(4, 2), E_3(16, 1), E_3(18, 1), E_3(24, 1), E_3(27, 1), E_3(30, 1), E_3(32, 1), E_3(36, 1), E_3(40, 1), E_3(48, 1) \).

By exhaustive search (as was done with \( k = 2 \)), all other 3-SS numbers can be found. They constitute the following set:

\[
\{42p | p = 7, 11, \ldots, 41\} \cup \{56p | p = 7, 11, \ldots, 43\} \cup \{64p | p = 2, 3, \ldots, 37\} \cup \\
\{726, 858, 250, 350, 225, 315, 968, 1144, 300, 420, 162, 270, 378, 243, 400, 560, 216, 360, 504, 324, 288, 480, 672, 432, 256, 384, 640, 576, 512, 768\}.
\]

None of the elements in the above sets are 3-SP, hence there are no 3-SP numbers. The diversity of possible 3-ascending sequences makes it difficult to rule out the existence of 3-cycles as we did 2-cycles. This is illustrated in the following example.

**Example 4:** If \( p_1, q_1 \) are odd primes, then \( s_3(4p_1q_1) = 4(p_1q_1 + p_1 + q_1) \). It is possible that \( p_1q_1 + p_1 + q_1 = p_2q_2 \), where \( p_2, q_2 \) are again odd primes, and so on. Several such sequences exist the longest one with \( p_1q_1 < 50000 \), and \( p_i, q_i > 3 \) is:

\[
\begin{align*}
184892 &= 4 \cdot 17 \cdot 2719 \xrightarrow{s_3} 195836 = 4 \cdot 173 \cdot 283 \xrightarrow{s_3} 197660 = 4 \cdot 5 \cdot 9883 \\
\xrightarrow{s_3} 237212 &= 4 \cdot 31 \cdot 1913 \xrightarrow{s_3} 244988 = 4 \cdot 73 \cdot 839 \xrightarrow{s_3} 248636 = 4 \cdot 61 \cdot 1019 \\
\xrightarrow{s_3} 252956 &= 4 \cdot 11 \cdot 5749 \xrightarrow{s_3} 275996 = 4 \cdot 7 \cdot 9857 \xrightarrow{s_3} 315452 = 4 \cdot 17 \cdot 4639 \\
\xrightarrow{s_3} 334076 &= 4 \cdot 47 \cdot 1777 \xrightarrow{s_3} 341372 = 4 \cdot 31 \cdot 2753 \xrightarrow{s_3} 352508 = 4 \cdot 13 \cdot 6779 \\
\xrightarrow{s_3} 379676 &= 4 \cdot 11 \cdot 8629 \xrightarrow{s_3} 414236 = 4 \cdot 29 \cdot 3571 \xrightarrow{s_3} 428636 = 4 \cdot 13 \cdot 8243 \\
\xrightarrow{s_3} 461660 &= 4 \cdot 5 \cdot 41 \cdot 563.
\end{align*}
\]
It seems highly unlikely, however, that a 3-ascending sequence be infinite. This is part of our final conjecture:

**Conjecture 2**: Any $k$-ascending sequence is finite.

**References**


[6] Hunsucker, J. L. and Pomerance, C., *There are no odd super perfect numbers less than $7 \times 10^{24}$*, Indian J. Math. 17 (1975), 107-120.


