ON CONSECUTIVE INTEGER PAIRS WITH THE SAME SUM OF DISTINCT PRIME DIVISORS

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Abstract

We define the arithmetic function $P$ by $P(1) = 0$, and $P(n) = p_1 + p_2 + \cdots + p_k$ if $n$ has the unique prime factorization given by $n = \prod_{i=1}^k p_i^{a_i}$; we also define $\omega(n) = k$ and $\omega(1) = 0$. We study pairs $(n, n+1)$ of consecutive integers such that $P(n) = P(n+1)$. We prove that $(5, 6)$, $(24, 25)$, and $(49, 50)$ are the only such pairs $(n, n+1)$ where $\{\omega(n), \omega(n+1)\} = \{1, 2\}$. We also show how to generate certain pairs of the form $(2^{2n}pq, rs)$, with $p < q$, $r < s$ odd primes, and lend support to a conjecture that infinitely many such pairs exist.

Keywords: Ruth–Aaron pairs, cyclotomic polynomials, Pell sequences, primes

Subject Class: 11A25, 11Y55

1. Introduction

For positive integers $n$, we define the arithmetic function $P(n)$ by $P(1) = 0$, and, for a positive integer $n$ having as its unique prime factorization $n = \prod_{i=1}^k p_i^{a_i}$, we have $P(n) = p_1 + p_2 + \cdots + p_k$. That is, $P(n)$ gives the sum of prime divisors of $n$ without multiplicity taken into account. The function is additive, in that $P(mn) = P(m) + P(n)$ if $(m, n) = 1$.

This function compares to the arithmetic function defined for positive integers $n$ by $S(1) = 0$ and $S(n) = \sum_{i=1}^k a_i p_i$ whenever $n = \prod_{i=1}^k p_i^{a_i}$; that is, $S(n)$ gives the sum of primes dividing $n$, taken with multiplicity. Then $S(n)$ is completely additive, in that $S(mn) = S(m) + S(n)$ for any two positive integers $m$ and $n$. A Ruth–Aaron pair is a pair $(n, n+1)$ of consecutive integers such that $S(n) = S(n+1)$. These were first discussed by Pomerance et. al. [4], and have been the subject of several articles (such as by Pomerance [6], Drost [2]) and numerous websites since.

However, in this article we are interested in finding pairs of consecutive positive integers...
(n, n + 1), such that \( P(n) = P(n + 1) \). For the sake of easy reference, we may call these *Ruth–Aaron pairs of the second kind*, or RAP2s for short. Note, however, that a RAP2 is also an ordinary Ruth–Aaron pair if both members \( n \) and \( n + 1 \) are square-free.

Some observations regarding RAP2s are immediate. For example, the members \((n, n + 1)\) of a RAP2 are of opposite parity, and are relatively prime. Let \( n \) be a positive integer. If \( n \) has the unique prime factorization \( n = \prod_{i=1}^{k} p_i^{a_i} \), then the prime powers \( p_i^{a_i} \), \( 1 \leq i \leq k \), are called the *components* of \( n \), and we define \( \omega(n) = k \), \( \omega(1) = 0 \) (thus \( \omega \) counts the components of \( n \)). For any given RAP2 \((n, n + 1)\), since 2 divides exactly one of the members (all other prime divisors of \( n \) and \( n + 1 \) being odd), we see that \( \omega(n) \) and \( \omega(n + 1) \) are of opposite parity.

In this article we shall completely determine all RAP2s \((n, n + 1)\) whose members have one or two components. We will also investigate RAP2s of the form \((2^{2n}pq, rs)\), with \( p < q \), \( r < s \) odd primes.

### 2. Preliminaries

If \( p \) is a prime and \( a, m \), are positive integers we write \( p^m \parallel a \) if \( p^m \mid a \) and \( p^{m+1} \nmid a \). In this case we say \( p^m \) *exactly* divides \( a \). For distinct primes \( p \) and \( q \) we write \( e^p(q) \) to denote the exponent to which \( q \) belongs modulo \( p \).

For positive integers \( n \), we denote the \( n \)th cyclotomic polynomial evaluated at \( x \) by \( \Phi_n(x) \). The cyclotomic polynomials (as shown by Niven [5], Ch. 3) may be defined inductively by

\[
x^n - 1 = \prod_{d \mid n} \Phi_d(x).
\]

By Theorems 94 and 95, Nagell [3], Ch. 5, we have

**Lemma 1.** Let \( p \) be and \( q \) be odd primes and let \( m \) be a positive integer. Let \( h = e_p(q) \). Then \( p \mid \Phi_m(q) \) if and only if \( m = hp^j \) for some integer \( j \geq 0 \). If \( j > 0 \) then \( p \parallel \Phi_{hp^j}(q) \).

**Lemma 2.** Let \( q \) be an odd prime and let \( m \) be a positive integer. Then \( 2 \mid \Phi_m(q) \) if and only if \( m = 2^j \) for some integer \( j \geq 0 \). If \( j > 1 \) then \( 2 \parallel \Phi_{2^j}(q) \).

Let \( q \) be prime and let \( m > 0 \) be an integer. Since, by definition,

\[
\Phi_m(q) = \prod_{k=1}^{m-1} (q - e^{2\pi i k/m}),
\]

and since \( \Phi_m(q) > 0 \), we have

\[
\Phi_m(q) = \prod_{k=1}^{m-1} \left| q - e^{2\pi i k/m} \right|,
\]

and since \( |q - e^{2\pi i k/m}| \geq q - 1 \) for \( 1 \leq k \leq m - 1 \), we have
Lemma 3. For a prime $q$ and an integer $m > 0$ we have $\Phi_m(q) \geq (q - 1)^{\phi(m)}$.

3. RAP2s of the form $(2^a p^b, q^c)$

The smallest numbers of components the members of a RAP2 can have are 1 and 2. In this instance, the members of the RAP2 have the form $2^a p^b$ and $q^c$ for positive integers $a$, $b$, and $c$, where $p$ and $q$ are necessarily twin (odd) primes (that is, $p + 2 = q$). We have two cases arising in this instance, those being $2^a p^b = q^c \pm 1$. In this section we consider the easier case of the two,

$$2^a p^b = q^c - 1. \tag{2}$$

Clearly $c > 1$, since $2^a p^b \geq 2(q - 2) = q + (q - 4) > q - 1$. Thus, since $q = p + 2$, (2) factors as

$$2^a p^b = (p + 1)(q^{c-1} + q^{c-2} + \cdots + q + 1).$$

Since $(p, p + 1) = 1$, it follows that $p + 1 = 2^t$ for some positive integer $t$. Hence $p = 2^t - 1$, $q = 2^t + 1$, which is possible only if $t = 2$; that is, $p = 3$ and $q = 5$. Then (2) becomes

$$2^a 3^b = 5^c - 1.$$

Since $5^c \equiv 1 \pmod{3}$, we have $2 \mid c$, so we write $c = 2\gamma$ for some positive integer $\gamma$. Thus

$$2^a 3^b = (5^\gamma - 1)(5^\gamma + 1).$$

Since $2 \mid 5^\gamma + 1$, we must have $3 \mid 5^\gamma + 1$. Since $(5^\gamma + 1, 5^\gamma - 1) = 2$, we have $3 \nmid 5^\gamma - 1$. Hence

$$5^\gamma - 1 = 2^{a-1}, \quad 5^\gamma + 1 = 2 \cdot 3^b.$$

Certainly $\gamma$ is odd (as $3 \nmid 5^\gamma - 1$). Suppose $\gamma > 1$. Then

$$5^\gamma - 1 = (5 - 1)(5^{\gamma-1} + 5^{\gamma-2} + \cdots + 5 + 1).$$

But the second factor is odd, and greater than 1; this contradicts $5^\gamma - 1 = 2^{a-1}$. Therefore $\gamma = 1$, and so $c = 2$. Hence (2) becomes $2^3 \cdot 3 = 5^2 - 1$; that is, $a = 3$, $b = 1$, and we have the RAP2 $(24, 25)$. Hence the only RAP2 of the form $(2^a p^b, q^c)$ is $(24, 25)$.

4. RAP2s of the form $(q^c, 2^a p^b)$

Suppose now that

$$2^a p^b = q^c + 1 \tag{3}$$

for positive integers $a$, $b$, and $c$, where $p$ and $q$ are primes such that $p + 2 = q$. This case is more difficult than that in Section 3.
By (1), we see that (3) is equivalent to

\[(4) \quad 2^ap^b = \prod_{d \mid 2c, d \mid c} \Phi(d).\]

Let \(h = e_p(q)\); we observe \(h = e_p(2)\) as well (since \(q = p + 2\)). By Lemmas 1 and 2, each divisor \(d \mid 2c\) such that \(d \nmid c\) must either have the form \(hp^j\) for some integer \(j \geq 0\) or the form \(2^k\) for some integer \(k \geq 1\). Writing \(c = 2^m s\) for some integer \(m \geq 0\) and odd integer \(s\), we see that \(2^{m+1} \parallel d\) for all divisors \(d \mid 2c\) such that \(d \nmid c\). In particular \(2^{m+1} \parallel h\).

Suppose \(s\) is composite. Then \(t \mid s\) for some odd integer \(t\) such that \(1 < t < s\). Then by (4), \(\Phi_s(q) \Phi_t(q) \parallel 2^ap^b\). This is impossible as \(2 \nmid \Phi_s(q) \Phi_t(q)\) by Lemma 2, and, as \(2 \mid h\), we have \(h \nmid s\), and so \(p \nmid \Phi_s(q) \Phi_t(q)\) by Lemma 1. Hence either \(s\) is prime or \(s = 1\).

Suppose \(s\) is prime. Then \(h = 2^{m+1} s\). For, if this were not the case then we would have \(h = 2^{m+1}\); since \(2 \parallel \Phi_2(q)\) by Lemma 2, it follows that either \(p \nmid \Phi_2(q)\) (if \(s \neq p\)) or \(p \parallel \Phi_2(q)\) (if \(s = p\)) by Lemma 1. The former possibility clearly contradicts (4); the latter implies \(\Phi_2(q) = p\), which is impossible as \(\Phi_2(q) > p\) by Lemma 3.

Therefore, since \(h = 2^{m+1} s\), we have

\[(5) \quad 2^ap^b = \Phi_{2^{m+1} s}(q) \Phi_{2m+1}(q)\]

by (4). This implies \(m = 0\) because otherwise (5) is impossible since we have \(p \nmid \Phi_{2^{m+1} s}(q), 2 \parallel \Phi_{2m+1}(q), \text{ and } \Phi_{2m+1}(q) > 2\) by Lemmas 1, 2, and 3 respectively. Therefore \(h = 2s\) and

\[2^ap^b = \Phi_2(q) \Phi_{2s}(q)\]

with \(2^a = \Phi_2(q) = q+1\) and \(p^b = \Phi_{2s}(q)\). But then \(q = 2^a - 1\), so that \(p = 2^a - 3\). It is clear that \(a > 2\), hence \(p \equiv 5 \pmod{8}\). Thus 2 is a quadratic nonresidue of \(p\), and hence \(2^{(p-1)/2} \equiv -1 \pmod{p}\) by Euler’s criterion. Since \(2^2 \mid p - 1 = 2^a - 4\), it follows that \(2^2 \mid e_p(2)\). But \(e_p(2) = h\), and since \(h = 2s\), we have \(2 \parallel h\), a contradiction.

Therefore \(s = 1\) and hence \(c = 2^m\) for some integer \(m \geq 0\). Thus (3) becomes

\[(6) \quad 2^ap^b = q^{2^m} + 1.\]

First let us suppose that \(m > 1\). Then \(q^{2^m} \equiv 1 \pmod{4}\) so that \(a = 1\). Hence

\[(7) \quad 2p^b = q^{2^m} + 1.\]

Since \(p \mid q^{2^m} + 1 = \Phi_{2m+1}(q)\), it follows from Lemma 1 that \(h = 2^{m+1}\); recalling as well \(h = e_p(2)\), it follows that \(p \equiv 1 \pmod{2^{m+1}}\). Since \(e_p(2) = 2^{m+1}\) and \(\Phi_{2m+1}(2) = 2^{2m} + 1\), it follows from Lemma 1 that

\[(8) \quad p \mid 2^{2^m} + 1.\]

Suppose \(p = 2^{m+1} t + 1\) for some odd integer \(t\). Then, as \(2^{2^m} \equiv -1 \pmod{p}\) by (8),

\[2^{(p-1)/2} = 2^{2^m t} = (2^{2^m})^t \equiv (-1)^t \equiv -1 \pmod{p},\]
and hence \((\frac{2}{p}) = -1\) by Euler’s criterion, where \((\frac{\cdot}{\cdot})\) denotes the Legendre symbol. But, \(p \equiv 1 (mod \ 8)\), which implies \((\frac{2}{p}) = +1\), a contradiction. Therefore

\[(9) \quad p \equiv 1 (mod \ 2^{m+2}). \]

Suppose \(b > 2^m\). Then from (7) we have

\[2p^{b-2^m} = \left(1 + \frac{2}{p}\right)^{2^m} + \frac{1}{p^{2^m}}.\]

By (9), \(p > 2^{m+2}\), and so

\[2p^{b-2^m} < \left(1 + \frac{1}{2^m}\right)^{2^m} + 1 < e + 1 < 4,\]

which implies \(2p < 4\), a contradiction. On the other hand, suppose \(b < 2^m\). Then by (6),

\[2 = \left(1 + \frac{2}{p}\right)^b (p + 2)^{2^m-b} + \frac{1}{p^b} > (p + 2)^{2^m-b} \geq p + 2,\]

a contradiction. Therefore we must have \(b = 2^m\), so that (7) becomes

\[(10) \quad 2p^{2^m} = q^{2^m} + 1.\]

Since \(q = p + 2\), (10) becomes

\[p^{2^m} = -p^{2^m} + (p + 2)^{2^m} + 1 = \sum_{k=1}^{2^m} \binom{2^m}{k} p^{2^m-k} 2^k + 1.\]

Since by (9) \(p > 2^{m+2}\), we have for each \(k\) such that \(1 \leq k \lesssim 2^m\),

\[\binom{2^m}{k} p^{2^m-k} 2^k = \frac{2^m(2^m-1)(2^m-2) \cdots (2^m-k+1)}{k!} \cdot p^{2^m-k} 2^k < \frac{2^m}{k!} \cdot p^{2^m-k} 2^k = \frac{1}{2^k} \cdot \frac{1}{k!} \cdot p^{2^m}.\]

Hence by (11),

\[p^{2^m} < \sum_{k=1}^{2^m} \frac{1}{2^k} \cdot \frac{1}{k!} \cdot p^{2^m} + 1 < p^{2^m} \left(\sqrt{e} - 1 + \frac{1}{p^{2^m}}\right) < 0.8p^{2^m},\]

a contradiction.

Hence we have (6) with either \(m = 0\) or \(m = 1\). If \(m = 0\) then (6) becomes

\[2^a p^b = q + 1 = p + 3,\]
implying \( p \mid 3 \), and hence \( p = 3, q = 5 \). Therefore \( 2^a3^b = 6 \), and so \( a = b = 1 \), and we have the RAP2 \((5,6)\).

If \( m = 1 \) then (6) becomes
\[
2^ap^b = q^2 + 1,
\]
which implies \( a = 1 \) since \( q^2 \equiv 1 \pmod{4} \). Therefore
\[
2p^b = q^2 + 1 = (p + 2)^2 + 1 = p^2 + 4p + 5,
\]
implying \( p \mid 5 \), and hence \( p = 5, q = 7 \). Therefore \( 2 \cdot 5^b = 50 \), and so \( b = 2 \), and we have the RAP2 \((49,50)\). Therefore the only RAP2s of the form \((q^c, 2^ap^b)\) are \((5,6)\) and \((49,50)\).

We summarize our results from this and the previous section:

**Theorem 1.** The only RAP2s \((n,n+1)\) such that \(\{\omega(n), \omega(n+1)\} = \{1, 2\} \) are \((5,6)\), \((24,25)\), and \((49,50)\).

5. RAP2s of the form \((2^{2n}pq, rs)\)

We now turn our attention to RAP2s \((n,n+1)\) where \(\{\omega(n), \omega(n+1)\} = \{2, 3\} \). There are 88 such pairs less than \(10^9\). Of these, 41 have the form \((4pq, rs)\), six have the form \((16pq, rs)\), and three have the form \((64pq, rs)\), with \(p < q, r < s \) odd primes. Among the remaining 38 pairs, no discernable patterns emerged. These data led us to narrow our investigation to those pairs of the form \((2^{2n}pq, rs)\), \(n \geq 1\).

Given such a pair, we have
\[
(12) \quad 2 + p + q = r + s, \\
(13) \quad 2^{2n}pq + 1 = rs.
\]

By (12) we have integers \(x, y, \) and \(z\) such that
\[
(14) \quad r = x - y, \quad s = x + y, \\
(15) \quad p = x - 1 - z, \quad q = x - 1 + z.
\]

Substituting (14) and (15) into (13), and simplifying, gives us
\[
((2^{2n} - 1)x - (2^{2n} + 1))(x - 1) = (2^n z - y)(2^n z + y),
\]
which may be expressed as
\[
(16) \quad \frac{(2^{2n} - 1)x - (2^{2n} + 1)}{2^n z - y} = \frac{2^n z + y}{x - 1} = a/b,
\]
where \(a/b\) represents the fractions in (16) in their lowest terms; thus \((a, b) = 1\). Separating the variables \(x, y, \) and \(z\) in (16) gives us
\[
(2^{2n} - 1)bx + ay - 2^n az = (2^{2n} + 1)b, \\
ax - by - 2^n bz = a,
\]
which we solve for $x, y$, in terms of $z$:

\[(a^2 + (2^n - 1)b^2)x = 2^{n+1}abz + a^2 + (2^n + 1)b^2,\]  

\[(a^2 + (2^n - 1)b^2)y = 2^n(a^2 - (2^n - 1)b^2)z + 2ab.\]

Our data of RAP2s less than $10^9$ revealed to us many different rational numbers for the quotient $a/b$ in (16), but some persisted more than others, especially $2/1$ and $7/4$ in the cases where $n = 1$. Recognizing these values as solutions to the Pell equation $a^2 - 3b^2 = 1$, we decided to assume that $a, b$, solved the Pell equation

\[a^2 - (2^n - 1)b^2 = 1\]

in the general case for $n \geq 1$. Under this hypothesis, (17) and (18) simplify to

\[(2a^2 - 1)x = 2^{n+1}abz + 2a^2 + 2b^2 - 1,\]

\[(2a^2 - 1)y = 2^n z + 2ab.\]

It is well known (e.g., as shown by Shockley [7], Ch. 12) that all positive solutions to (19) are given by

\[a_1 = 2^n, \quad b_1 = 1,\]

\[a_{j+1} = 2^n a_j + (2^n - 1)b_j \quad (j \geq 1),\]

\[b_{j+1} = a_j + 2^n b_j \quad (j \geq 1).\]

One shows by induction that $2^n \mid a_j b_j$ for all $j \geq 1$. Hence we may parametrize $z$ from (20): since $y$ is an integer it follows that $2a^2 - 1$ divides $2^n z + 2ab$, and since $2a^2 - 1$ is odd we have

\[z \equiv -\frac{2ab}{2^n} \pmod{2a^2 - 1}.\]

Thus $z$ has the form given by

\[z = (2a^2 - 1)k + 2a^2 - 1 - \frac{2ab}{2^n}.\]

for integers $k \geq 0$. Substituting (22) into (17) and (18) gives us

\[x = 2^{n+1}abk + 2^{n+1}ab - 2b^2 + 1,\]

\[y = 2^n k + 2^n.\]

Substituting (22), (23), and (24) into (14) and (15) gives us

**Theorem 2.** Let integral $n \geq 1$ be given and let $a, b$, be solutions to the Pell equation (19). Then $(2^{2n}pq, rs)$ is a RAP2 if, for an integer $k \geq 0$, the following four quantities are all prime:

\[p = 2(2^{n+1}ab - 2a^2 + 1)k + \left(2^{n+1} - 2b^2 - 2a^2 + 1 + \frac{2ab}{2^n}\right),\]

\[q = 2(2^{n+1}ab + 2a^2 - 1)k + \left(2^{n+1} - 2b^2 + 2a^2 - 1 - \frac{2ab}{2^n}\right),\]

\[r = 2^{n+1}(2ab - 1)k + 2^n(2ab - 1) - 2b^2 + 1,\]

\[s = 2^{n+1}(2ab + 1)k + 2^n(2ab + 1) - 2b^2 + 1.\]
Note that we substituted $2k$ instead of $k$ to ensure $p$ and $q$ as given in Theorem 2 are odd. We also kept $2ab$ in the numerators above (rather than reduce to $ab/2^{a-1}$) since the Pell sequences $(21)$ have the property $b_{2j} = 2a_j b_j$ (as well as $a_{2j} = 2a_j^2 - 1$). Moreover, one shows by induction that for all $n$ and $k$, if $a_{3j}, b_{3j}$ in $(21)$ are the solutions used in applying Theorem 2, then at least one of $p, q, r, s$ is divisible by 3 (hence no RAP2 is produced).

There are 149 RAP2s of the form $(2^{2n}pq, rs)$ whose elements are less than $2^{34}$. Of these, 116 correspond to $n = 1$, and 16 of these involve the solutions $a_1 = 2$, $b_1 = 1$ of the Pell equation $a^2 - 3b^2 = 1$, while an additional 3 involve $a_2 = 7$, $b_2 = 4$. Also, 16 such RAP2s correspond to $n = 2$, 3 of which involve the solutions $a_1 = 4$, $b_1 = 1$ of the equation $a^2 - 15b^2 = 1$, and 9 of the RAP2s correspond to $n = 3$, 3 of which involve $a_1 = 8$, $b_1 = 1$ ($a^2 - 63b^2 = 1$). Finally, 3 of the RAP2s involve $n = 4$.

We had found the RAP2s less than $2^{34}$ by a straightforward computer search. Later on, we applied Theorem 2 to search for the RAP2s of the special form described in that theorem. We found literally thousands of them. We computed them on a PC, using the UBASIC software package. Primality of $p, q, r, s$, were verified by the APR primality test due to Adleman, Pomerance, and Rumely [1].

6. Concluding Remarks

It is unknown if there are infinitely many RAP2s. The question of infinitude also remains open for ordinary Ruth–Aaron pairs—see Pomerance [6] for a detailed history. In light of Theorem 2, fixing $n$ at say $n = 1$, if one could show that for each solution $a_j, b_j, 3 \nmid j$, to (19), there exists at least one $k$ for which $p, q, r, s$, are all prime, then a proof of infinitely many RAP2s of the form $(4pq, rs)$ would be obtained. We have not been able to produce such a proof, but we conjecture the existence of infinitely many RAP2s nonetheless.

We have also considered RAP2s $(n, n + 1)$ for which $\{\omega(n), \omega(n + 1)\} = \{1, 4\}$. These would be obtained by finding distinct odd primes $p_1, p_2, p_3$, and positive integers $a, b_1, b_2, b_3, c$, such that

$$2 + p_1 + p_2 + p_3 = q,$$

and such that

$$2^a p_1^{b_1} p_2^{b_2} p_3^{b_3} = q^c \pm 1.$$  

(25)

Let $h = [e_{p_1}(q), e_{p_2}(q), e_{p_3}(q)]$. Then $p_1, p_2, p_3$, all divide $q^c - 1$ only if $h \mid c$, in which case $q^h - 1$ divides $2^a p_1^{b_1} p_2^{b_2} p_3^{b_3}$. Thus if $q^h - 1$ is found to contain any prime factors other than $2, p_1, p_2, p_3$, then a contradiction is obtained. Using modular arithmetic, we can find $\alpha, \beta_1, \beta_2, \beta_3$, such that $2^a \| q^h - 1$ and $p_i^{\beta_i} \| q^h - 1$ ($1 \leq i \leq 3$). A contradiction is obtained if $2^a p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} < q^h - 1$.

In the case of $q^c + 1$, (26) becomes

$$2^a p_1^{b_1} p_2^{b_2} p_3^{b_3} = \prod_{d \mid 2c, d \mid c} \Phi_d(q)$$

(26)

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by (1). By Lemma 1, the primes $p_1$, $p_2$, $p_3$, all divide $q^c + 1$ only if $e_{p_1}(q)$, $e_{p_2}(q)$, and $e_{p_3}(q)$ are all even such that each quantity is exactly divisible by the same power of 2. In this case we have $q^{b/2} + 1 | 2^{a} p_1^{b_1} p_2^{b_2} p_3^{b_3}$. Thus a contradiction is obtained if $q^{b/2} + 1$ contains any prime factors other than 2, $p_1$, $p_2$, or $p_3$.

For all odd primes $q < 20000$, we found all triples of odd primes $p_1 < p_2 < p_3$ satisfying (25), and then we disproved the possibility of (25) and (26) by computation. We conjecture the nonexistence of RAP2s $(n, n + 1)$ for which $\{\omega(n), \omega(n + 1)\} = \{1, 4\}$, although we have not yet obtained a proof.

References


