ON 2-ADIC ORDERS OF STIRLING NUMBERS OF THE SECOND KIND

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Abstract

We prove that for any $k = 1, \ldots, 2^n$ the 2-adic order of the Stirling number $S(2^n, k)$ of the second kind is exactly $d(k) - 1$, where $d(k)$ denotes the number of 1’s among the binary digits of $k$. This confirms a conjecture of Lengyel.

1. Introduction

For a nonzero integer $m$, if $2^h$ is the highest power of two dividing $m$, then we say that the 2-adic order $\rho_2(m)$ of $m$ is $h$. In this paper $\rho_2(\cdot)$ is called the 2-adic valuation function.

Legendre observed that if $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ then $\rho_2(n!) = n - d(n)$, where $d(n)$ is the number of 1’s in the binary representation of $n$, in other words $d(n) = \sum_{\lambda=0}^{\infty} \varepsilon_\lambda(n)$ if $n = \sum_{\lambda=0}^{\infty} \varepsilon_\lambda(n) 2^\lambda$ with $\varepsilon_\lambda(n) \in \{0, 1\}$. Kummer proved that $\rho_2 \left( \binom{n}{k} \right) = d(k) + d(n-k) - d(n)$ whenever $0 \leq k \leq n$.

Let $n \in \mathbb{N}$. The Stirling numbers $S(n, k) \ (k \in \mathbb{N})$ of the second kind are given by

$$x^n = \sum_{k=0}^{\infty} S(n, k) (x)_k,$$

where $(x)_k = x(x-1)(x-2)\ldots(x-k+1)$ for $k \in \mathbb{N} \setminus \{0\}$ and $(x)_0 = 1$. Actually $S(n, k)$ is the number of ways in which it is possible to partition a set with $n$ elements into exactly $k$ nonempty subsets. For more details and basic results on Stirling numbers of the second kind we refer the reader to [2] and [4].

In this paper we study 2-adic orders of Stirling numbers of the second kind, and establish the following theorem which was conjectured by T.Lengyel [3] and verified by him in some special cases.

**Theorem 1.** Let $n, k \in \mathbb{N}$ and $1 \leq k \leq 2^n$. Then we have

$$\rho_2(S(2^n, k)) = d(k) - 1.$$
In the next section we reveal some useful properties of Stirling numbers of the second kind. We are going to prove Theorem 1 in Section 3 on the basis of Section 2.

2. Auxiliary results on Stirling numbers of the second kind

The following identity relates the Stirling numbers of the second kind $S(n+m, \cdot)$ to $S(n, \cdot)$ and $S(m, \cdot)$.

**Theorem 2.** Let $n, m, k \in \mathbb{N}$ such that $0 \leq k \leq n + m$. Then

$$S(n+m, k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(n, k-i) S(m, j).$$

**Proof.** Let $n, m \in \mathbb{N}$. Then

$$x^{n+m} = x^n x^m = \sum_{r=0}^{n} S(n, r)(x) \sum_{j=0}^{m} S(m, j)(x)_j$$

$$= \sum_{r=0}^{n} S(n, r)(x) \sum_{j=0}^{m} j! S(m, j) \left(\frac{x}{j}\right)$$

$$= \sum_{r=0}^{n} S(n, r)(x) \sum_{j=0}^{m} j! S(m, j) \sum_{i=0}^{j} \binom{x-r}{i} \binom{r}{j-i}$$

(by the Chu-Vandermonde identity)

$$= \sum_{r=0}^{n} S(n, r) \sum_{j=0}^{m} S(m, j) \sum_{i=0}^{j} \frac{j!}{i!} \binom{r}{j-i} (x)_{r+i}$$

Thus, for any $k = 0, 1, \ldots, n+m$ we have

$$S(n+m, k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \frac{j!}{i!} \binom{k-i}{j-i} S(n, k-i) S(m, j)$$

$$= \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(n, k-i) S(m, j).$$

**Remark:** Stirling numbers of the second kind occur in a natural way while making calculations in the Witt ring (see [1] for further details). It was in this context that the previous identity arose.
Lemma 1. Let $m, n \in \mathbb{N}$. Then
\[ d(m + n) \leq d(m) + d(n) \]
and equality holds if and only if
\[ \sum_{\lambda=0}^{\infty} \varepsilon_{\lambda}(m) \varepsilon_{\lambda}(n) = 0, \]
i.e., when $m$ and $n$ have no non-zero binary digit in common.

Proof. If $m$ and $n$ have no non-zero binary digit in common then it is obvious that $d(m+n) = \sum \varepsilon_{\lambda}(m+n) = \sum (\varepsilon_{\lambda}(m) + \varepsilon_{\lambda}(n)) = d(m) + d(n)$. On the other hand, suppose that $m$ and $n$ have a non-zero binary digit in common. Let us say that $\lambda_0$ is the lowest natural number such that $\varepsilon_{\lambda_0}(m) = \varepsilon_{\lambda_0}(n) = 1$. Then it is clear that $\varepsilon_{\lambda_0}(m+n) = 0$ and 1 is added to $\varepsilon_{\lambda_0+1}(m) + \varepsilon_{\lambda_0+1}(n)$ to obtain an expression for $\varepsilon_{\lambda_0+1}(m+n)$. Anyhow, at least one non-zero binary digit is lost in $d(m+n)$.

Remark: The case $d(m+n) = d(m) + d(n) - 1$ occurs if and only if $\varepsilon_{\lambda_0+1}(m) = \varepsilon_{\lambda_0+1}(n) = 0$ with $\lambda_0$ the unique natural number such that $\varepsilon_{\lambda_0}(m) = \varepsilon_{\lambda_0}(n) = 1$.

A new lower bound on the 2-adic order of Stirling numbers of the second kind can be obtained as follows.

Theorem 3. Let $n, k \in \mathbb{N}$ and $0 \leq k \leq n$. Then
\[ \rho_2(S(n,k)) \geq d(k) - d(n). \]

Proof. We use induction on $n$.

For $n = 0$, $\rho_2(S(0,0)) = \rho_2(1) \geq d(0) - d(0)$.

Assume now that the above inequality is true for all $i < n$. We will prove the theorem for $n$. Observe that for $k = 0$ the result is obviously true.

Let $1 \leq k \leq n$. The Stirling numbers of the second kind satisfy the well-known ‘vertical’ recurrence relation
\[ S(n,k) = \sum_{i=k-1}^{n-1} \binom{n-1}{i} S(i, k-1). \]

Combining this with the ‘triangular’ recurrence relation
\[ S(n,k) = S(n-1,k-1) + kS(n-1,k) \]
we obtain
\[ kS(n, k) = \sum_{i=k-1}^{n-1} \binom{n}{i} S(i, k - 1). \]

Thus
\[ \rho_2(kS(n, k)) = \rho_2 \left( \sum_{i=k-1}^{n-1} \binom{n}{i} S(i, k - 1) \right) \]
\[ \geq \min_{k-1 \leq i \leq n-1} \{ \rho_2 \left( \binom{n}{i} \right) + d(k-1) - d(i) \} \] (by the induction hypothesis)
\[ = \min_{k-1 \leq i \leq n-1} \{ d(n - i) + d(k-1) - d(n) \} \] (by the Kummer identity)
\[ = d(k - 1) - d(n) + 1. \]

So,
\[ \rho_2(S(n, k)) \geq d(k - 1) - \rho_2(k) + 1 - d(n) \]
\[ = d(k) - d(n). \]

3. Proof of Lengyel’s conjecture

We use induction on \( n \). For \( n = 0 \), \( \rho_2(S(1, 1)) = \rho_2(1) = 0 = d(1) - 1 \). We assume the theorem is true for all powers \( 2^i \) where \( i < n \). We will prove that the theorem holds for \( 2^n \).

By Theorem 2
\[ S(2^n, k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k - i)!}{(k - j)!} S(2^{n-1}, k - i)S(2^{n-1}, j). \] (1)
We will take a closer look at the 2-adic valuation of each term in this sum (1).

\[
\rho_2 \left( \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(2^{n-1}, k-i)S(2^{n-1}, j) \right) 
\]

\[
= \rho_2 \left( \binom{j}{i} \right) + \rho_2((k-i)! + \rho_2((k-j)! + \rho_2(S(2^{n-1}, k-i)) + \rho_2(S(2^{n-1}, j)) 
\]

\[
= \rho_2 \left( \binom{j}{i} \right) + \rho_2((k-i)! - \rho_2((k-j)! + d(k-i) + d(j) - 2 
\]

(by the induction hypothesis)

\[
= d(i) + d(j-i) - d(j) + (k-i) - d(k-i) - (k-j) + d(k-j) + d(k-i) + d(j) - 2 
\]

(by the Kummer and Legendre identities)

\[
= d(i) + d(j-i) + i + d(k-j) - 2. 
\]

The inequality of Lemma 1 implies that

\[
d(i) + d(j-i) + j - i + d(k-j) - 2 \geq d(j) + i - d(k-j) - 2 \geq d(k) - 2 + j - i. 
\]

Since \( j \geq i \), the 2-adic valuation of every term in the sum is at least \( d(k) - 2 \). To prove that the 2-adic valuation of the global sum (1) equals \( d(k) - 1 \) we will calculate the number of terms with 2-adic valuation \( d(k) - 2 \) and the number of terms with 2-adic valuation \( d(k) - 1 \). These two results together will show that the 2-adic valuation of (1) equals \( d(k) - 1 \).

For \( k = 1 \) the theorem holds since \( \rho_2(S(2^n, 1)) = \rho_2(1) = 0 = d(1) - 1 \), for all \( n \in \mathbb{N} \). So assume \( k \neq 1 \).

**Case 1:** \( d(i) + d(j-i) + j - i + d(k-j) - 2 = d(k) - 2 \).

Since \( d(i) + d(j-i) + d(k-j) \geq d(k) \) and \( j \geq i \), this situation can occur only when \( j = i \) and \( d(i) + d(k-i) = d(k) \). By Lemma 1 this holds only when \( i \) and \( k - i \) have no non-zero binary digit in common, or equivalently, when \( \varepsilon_\lambda(i) + \varepsilon_\lambda(k-i) = \varepsilon_\lambda(k) \), for all \( \lambda \in \mathbb{N} \). If \( \varepsilon_\lambda(k) = 1 \) (this occurs \( d(k) \) times), the possible values for \( \varepsilon_\lambda(i) \) are 0 and 1.

If \( \varepsilon_\lambda(k) = 0 \), then \( \varepsilon_\lambda(i) = 0 \) as well.

So, for a given \( k \), there are \( 2^{d(k)} \) possibilities for \( i = j \). We need to modify this number of possibilities since it includes the non-occurring situations \( i = j = 0 \) and \( i = j = k \). This means we have \( 2^{d(k)} - 2 \) terms in (1) with 2-adic valuation \( d(k) - 2 \).

In the case where \( d(k) = 1 \), i.e. \( k = 2^m \), there are no terms satisfying the condition. When \( d(k) > 1 \), these \( 2^{d(k)} - 2 \) terms contribute, in total, \( M2^{d(k)-1} \) to (1).

We will show that \( M \) is odd. Let \( O(i) \) be the odd part of \( S(2^{n-1}, i) \). Consider the sum in this case
\[
\sum_{i=1}^{k-1} S(2^{n-1}, k-i) S(2^{n-1}, i) = \sum_{i=1}^{k-1} O(k-i)O(i)2^{d(k)-2}.
\]

The latter expression is invariant under switching \(i\) and \(k-i\) and since \(i = k/2\) (in the case \(k\) even) never occurs \((d(k/2) + d(k/2) = 2d(k) \neq d(k))\) we obtain
\[
\sum_{i=1}^{k-1} O(k-i)O(i)2^{d(k)-1}.
\]

This last expression consists of an odd number, \(2^{d(k)-1} - 1\), of terms, so it contributes, in total, \(M2^{d(k)-1}\) to (1), where \(M\) is odd.

**Case 2**: \(d(i) + d(j - i) + j - i + d(k - j) - 2 = d(k) - 1\).

Since \(d(i) + d(j - i) + d(k - j) \geq d(k)\) and \(j \geq i\), this situation can occur only when \(j = i + 1\) and \(d(i) + d(k - i - 1) = d(k) - 1\) or when \(j = i\) and \(d(i) + d(k - i) = d(k) + 1\).

**Case 2.1**: \(d(i) + d(k - i - 1) = d(k) - 1\) and \(j = i + 1\).

Since \(d(k - 1) \leq d(i) + d(k - i - 1) = d(k) - 1\), \(k\) must be odd. We have \(d(i) + d((k - 1) - i) = d(k - 1)\). As in Case 1, there are \(2^{d(k-1)}\) possible values for \(i\) (the case \(i = k\) doesn’t occur and the case \(i = 0\) and \(j = 1\) is allowed). This is an even number of terms since \(k \neq 1\).

**Case 2.2**: \(d(i) + d(k - i) = d(k) + 1\) and \(j = i\).

By Lemma 1 this can occur only when there is just one value of \(\lambda \in \mathbb{N}\) for which \(\varepsilon_\lambda(i) = \varepsilon_\lambda(k-i) = 1\). Moreover one must have \(\varepsilon_{\lambda+1}(i) = \varepsilon_{\lambda+1}(k-i) = 0\). This implies that \(\varepsilon_\lambda(k) = 0\) and \(\varepsilon_{\lambda+1}(k) = 1\). Following the same reasoning as in Case 1 with the remaining \(d(k) - 1\) non-zero binary digits of \(k\), we have \(2^{d(k)-1}\) possibilities for \(i\) (the cases \(i = 0\) and \(i = k\) don’t occur).

So there are \(2^{d(k)-1}\) terms in (1) which come under Case 2.2 (and thus have 2-adic valuation \(d(k) - 1\)). When \(d(k) = 1\), this number is 1, otherwise it is even.

After considering all the possible cases and counting the number of terms with 2-adic valuation \(d(k) - 2\) and 2-adic valuation \(d(k) - 1\), we can conclude that \(\rho_2(S(2^n, k)) = d(k) - 1\).
An overview of all the cases is given in the following table.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2.1</th>
<th>Case 2.2</th>
<th>coefficient of $2^{d(k)-1}$</th>
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<td>coefficient of $2^{d(k)-1}$</td>
<td>coefficient of $2^{d(k)-1}$</td>
<td>coefficient of $2^{d(k)-1}$</td>
</tr>
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<td>0</td>
<td>odd</td>
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<tr>
<td>$d(k) &gt; 1$ &amp; $k$ odd</td>
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<td>even</td>
<td>odd</td>
</tr>
<tr>
<td>$d(k) &gt; 1$ &amp; $k$ even</td>
<td>even</td>
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References


