A BIJECTIVE PROOF OF \( f_{n+4} + f_1 + 2f_2 + \cdots + nf_n = (n + 1)f_{n+2} + 3 \)

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Received: 10/6/05, Revised: 1/5/06, Accepted: 1/16/06, Published: 2/1/06

Abstract

In Proofs that Really Count, Benjamin and Quinn mentioned that there was no known bijective proof for the identity \( f_1 + 2f_2 + \cdots + nf_n = (n + 1)f_{n+2} - f_{n+4} + 3 \) for \( n \geq 0 \), where \( f_k \) is the \( k \)-th Fibonacci number. In this paper, we interpret \( f_k \) as the cardinality of the set \( F_k \) consisting of all ordered lists of 1's and 2's whose sum is \( k \). We then demonstrate a bijection between the sets \( F_{n+4} \cup \bigcup_{k=1}^{n} ([1,2,\ldots,k] \times F_k) \) and \( \{1, 2, \ldots, n + 1 \} \times F_{n+2} \) \( \cup \{1, 2, 3\} \), which gives a bijective proof of the identity.

1. Introduction

We will interpret the \( k \)-th Fibonacci number \( f_k \) as the cardinality of the set \( F_k \) of all ordered lists of 1's and 2's that have sum \( k \). Thus, \((f_0, f_1, f_2, f_3, f_4, f_5, \ldots) = (1, 1, 2, 3, 5, 8, \ldots)\). For an integer \( m \), the number \( mf_k \) will be interpreted as the cardinality of the Cartesian product \([m] \times F_k\), where \([m] := \{1, 2, 3, \ldots, m\}\).

On page 14 of Proofs that Really Count [1], Benjamin and Quinn mentioned that there was no known bijective proof for the identity \( f_1 + 2f_2 + \cdots + nf_n = (n + 1)f_{n+2} - f_{n+4} + 3 \) for \( n \geq 0 \). In Section 2 we define a map

\[ \phi : F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_k) \longrightarrow \{1, 2, 3\} \cup ([n+1] \times F_{n+2}), \]

and in Section 3 we describe why \( \phi \) is a bijection. This provides a bijective proof of the identity \( f_{n+4} + f_1 + 2f_2 + \cdots + nf_n = (n + 1)f_{n+2} + 3 \) for \( n \geq 0 \). For completeness, we also define the inverse map

\[ \psi : \{1, 2, 3\} \cup ([n+1] \times F_{n+2}) \longrightarrow F_{n+4} \bigcup \bigcup_{k=1}^{n} ([k] \times F_k) \]

in Section 4, and the cases in the definition of \( \psi \) correspond to those for \( \phi \).
When examples are given below, ordered lists are denoted using angled brackets, e.g., \( \langle a_1, a_2, \ldots, a_m \rangle \). Also, Doron Zeilberger [4] has written a Maple package that implements the bijection, which may be downloaded from

\[ \text{http://www.math.rutgers.edu/~zeilberg/tokhniot/PHIIL} \quad (1) \]

2. The bijection \( \phi \)

In this section we define the bijection

\[
\phi : F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_k) \longrightarrow \{1, 2, 3\} \cup ([n + 1] \times F_{n+2}).
\]

For \( X \in F_{n+4} \) where the last \( n \) numbers in the list \( X \) are 1’s, define \( \phi \) according to the chart below.

\[
\begin{align*}
\phi : \langle 1, 1, 1, 1, \overbrace{1, \ldots, 1}^{n} \rangle & \mapsto (1, \langle 1, 1, \ldots, 1 \rangle) \\
\phi : \langle 2, 1, 1, \overbrace{1, \ldots, 1}^{n} \rangle & \mapsto 2 \\
\phi : \langle 1, 2, 1, \overbrace{1, \ldots, 1}^{n} \rangle & \mapsto 3 \\
\phi : \langle 1, 1, 2, \overbrace{1, \ldots, 1}^{n} \rangle & \mapsto (1, \langle 2, 1, \ldots, 1 \rangle)
\end{align*}
\]

For all cases not covered by the chart above, we define \( \phi \) by the two cases below.

**Case 1:** Consider \( X \in F_{n+4} \), so \( X \) is a list of 1’s and 2’s that sums to \( n + 4 \). By the above special cases above, we know that \( X \) ends in a string of exactly \( \ell \) 1’s, where \( 0 \leq \ell < n \) (so \( X \) has a 2 followed by \( \ell \) 1’s at the end). Take \( X \) and delete the last 2 in \( X \) to get \( \tilde{X} \), which is an element of \( F_{n+2} \), and define \( \phi : X \mapsto (n - \ell + 1, \tilde{X}) \).

Examples for \( n = 3 \):

\[
\begin{align*}
\phi : \langle 1, 1, 2, 1, 2 \rangle & \mapsto (4, \langle 1, 1, 2, 1 \rangle) \\
\phi : \langle 1, 1, 1, 2, 1 \rangle & \mapsto (3, \langle 1, 1, 2, 1 \rangle) \\
\phi : \langle 1, 1, 2, 1, 1 \rangle & \mapsto (2, \langle 1, 1, 1, 1 \rangle)
\end{align*}
\]

**Case 2:** Consider \((i, X)\) where \( X \in F_k \) and \( i \in [k] \) (and thus \( i \leq k \)). Take \( X \) and append a 2 followed by \( (n - k) \) 1’s to get \( \tilde{X} \), which is an element of \( F_{n+2} \), and define \( \phi : (i, X) \mapsto (i, \tilde{X}) \).

Examples for \( n = 3 \):

\[
\begin{align*}
\phi : (1, (1)) & \mapsto (1, (1, 2, 1, 1)) \\
\phi : (1, (1, 1)) & \mapsto (1, (1, 1, 2, 1)) \\
\phi : (2, (2)) & \mapsto (2, (2, 2, 1)) \\
\phi : (2, (2, 1)) & \mapsto (2, (2, 1, 2))
\end{align*}
\]
3. Showing $\phi$ is bijective

The following three facts (which may be easily verified) help show that $\phi$ is injective:

1. The image of $\phi$ from the five special cases consists of $\{1, 2, 3\}$ and all elements $(i, Y)$ of $[n+1] \times F_{n+2}$ where $i = 1$ and $Y$ ends in at least $n = (n + 1 - i)$ 1’s.

2. The image of $\phi$ from Case 1 consists of all elements $(i, Y)$ of $[n+1] \times F_{n+2}$ where $2 \leq i \leq n + 1$ and $Y$ ends in at least $(n + 1 - i)$ 1’s.

3. The image of $\phi$ from Case 2 consists of all elements $(i, Y)$ of $[n+1] \times F_{n+2}$ where $1 \leq i \leq n$ and one of the last $(n + 1 - i)$ entries in $Y$ is a 2.

It is easily seen from the definition that $\phi$ restricted to Case 1 is injective; and similarly, $\phi$ is injective when restricted to Case 2 or to the five special cases. Thus, since the three images described above are distinct, $\phi$ as a whole is injective. Furthermore, the union of the three images above consists of all of $\{1, 2, 3\} \cup ([n+1] \times F_{n+2})$ (note that there is no element $(i, Y) \in [n+1] \times F_{n+2}$ with $i = n + 1$ and $Y$ containing a 2 in the last $(n + 1 - i)$ entries). Thus $\phi$ is a bijection.

4. The inverse bijection $\psi$

In this section we define the inverse bijection

$$
\psi : \{1, 2, 3\} \cup ([n+1] \times F_{n+2}) \longrightarrow F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_{k}).
$$

For elements of $\{1, 2, 3\}$ and for the elements $(1, \langle 1,1,\ldots,1 \rangle)$ and $(1, \langle 2,1,1,\ldots,1 \rangle)$ of $[n+1] \times F_{n+2}$, define $\psi$ according to the chart below.

$$
\begin{align*}
\psi : \langle 1,1,\ldots,1 \rangle & \mapsto \langle 1,1,1,1,\ldots,1 \rangle \\
\psi : 2 & \mapsto \langle 1,2,1,1,\ldots,1 \rangle \\
\psi : 3 & \mapsto \langle 1,1,2,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 2,1,1,\ldots,1 \rangle) & \mapsto \langle 2,2,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 1,1,\ldots,1 \rangle) & \mapsto \langle 1,1,1,1,\ldots,1 \rangle \\
\psi : 1 & \mapsto \langle 2,1,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 1,1,\ldots,1 \rangle) & \mapsto \langle 1,1,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 2,1,1,\ldots,1 \rangle) & \mapsto \langle 2,2,1,1,\ldots,1 \rangle \\
\psi : 3 & \mapsto \langle 1,1,2,1,1,\ldots,1 \rangle \\
\psi : 2 & \mapsto \langle 1,2,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 1,1,\ldots,1 \rangle) & \mapsto \langle 1,1,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 2,1,1,\ldots,1 \rangle) & \mapsto \langle 2,2,1,1,\ldots,1 \rangle \\
\psi : 1 & \mapsto \langle 2,1,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 1,1,\ldots,1 \rangle) & \mapsto \langle 1,1,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 2,1,1,\ldots,1 \rangle) & \mapsto \langle 2,2,1,1,\ldots,1 \rangle \\
\psi : 3 & \mapsto \langle 1,1,2,1,1,\ldots,1 \rangle \\
\psi : 2 & \mapsto \langle 1,2,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 1,1,\ldots,1 \rangle) & \mapsto \langle 1,1,1,1,\ldots,1 \rangle \\
\psi : (1, \langle 2,1,1,\ldots,1 \rangle) & \mapsto \langle 2,2,1,1,\ldots,1 \rangle \\
\psi : 1 & \mapsto \langle 2,1,1,1,\ldots,1 \rangle
\end{align*}
$$
For all cases not covered by the chart above, we define $\psi$ as follows. Consider $(i, Y)$, where $Y \in F_{n+2}$ and $i \in [n + 1]$.

**Case 1:** If $Y$ ends with at least $(n + 1 - i)$ 1’s, then insert a 2 before the last $(n + 1 - i)$ 1’s to get $\tilde{Y}$ and define $\psi : (i, Y) \mapsto \tilde{Y}$.

Examples for $n = 3$:

- $\psi : (4, \langle 1, 1, 1, 2 \rangle) \mapsto \langle 1, 1, 1, 2, 2 \rangle$
- $\psi : (3, \langle 2, 1, 1, 1 \rangle) \mapsto \langle 2, 1, 1, 2, 1 \rangle$
- $\psi : (2, \langle 1, 1, 1, 1 \rangle) \mapsto \langle 1, 1, 1, 2, 1, 1 \rangle$

**Case 2:** If one of the last $n + 1 - i$ entries in $Y$ is a 2, then delete the last 2 in $Y$ and all 1’s following that 2 to get $\hat{Y}$. Define $\psi : (i, Y) \mapsto (i, \hat{Y})$.

Examples for $n = 3$:

- $\psi : (1, \langle 1, 2, 1, 1 \rangle) \mapsto \langle 1, \langle 1 \rangle \rangle$
- $\psi : (1, \langle 1, 1, 2, 1 \rangle) \mapsto \langle 1, \langle 1, 1 \rangle \rangle$
- $\psi : (2, \langle 2, 2, 1 \rangle) \mapsto \langle 2, \langle 2 \rangle \rangle$
- $\psi : (2, \langle 2, 1, 2 \rangle) \mapsto \langle 2, \langle 2, 1 \rangle \rangle$

**Acknowledgements** The author would like to thank Doron Zeilberger for suggesting this problem and for providing valuable support in solving it. Thanks also is due to the anonymous referee for helpful comments in revising and polishing this paper.

**References**


