TRIANGULAR NUMBERS IN GEOMETRIC PROGRESSION

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Abstract

In [R. K. Guy, Unsolved Problems in Number Theory, 3rd ed. Springer Verlag, New York, 2004, D23], it is stated that Sierpinski asked the question of whether or not there exist four (distinct) triangular numbers in geometric progression. Szymiczek conjectured that the answer is negative. Recently M. A. Bennett [Integers: Electronic Journal of Combinatorial Number Theory 5(1) (2005)] proved that there do not exist four distinct triangular numbers in geometric progression with the common ratio being a positive integer. In this paper we prove that there do not exist four distinct triangular numbers in geometric progression. Thus Sierpinsk’s question is answered and Szymiczek’s conjecture is confirmed.

In [4, D23], it is stated that Sierpinski asked the question of whether or not there exist four (distinct) triangular numbers in geometric progression. Szymiczek conjectured that the answer is negative. Recall that a triangular number is one of the form $T_n = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$. The problem of finding three such triangular numbers is readily reduced to finding solutions to a Pell equation whereby, an old result of Gerardin[3] (see also[2], [5]) implies that there are infinitely many such triples, the smallest of which is $(T_1, T_3, T_5)$. Recently M. A. Bennett[1] proved that there do not exist four distinct triangular numbers in geometric progression with the ratio being positive integer. In this paper, we extend Bennett’s result to the rational common ratio and prove that there do not exist four distinct triangular numbers in geometric progression. Thus Sierpinsk’s question is answered and Szymiczek’s conjecture is confirmed.

Theorem There do not exist four distinct triangular numbers in geometric progression.

Proof. Suppose that there exist four distinct triangular numbers $T_{n_1}, T_{n_2}, T_{n_3}, T_{n_4}$ in geometric progression. Let $q$ be the common ratio. It is obvious that $q > 0$ and $q \neq 1$. Without

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loss of generality, we may assume that $0 < q < 1$. Let $a = 8T_{n_4}$. Then

$$8T_{n_2} = aq, \quad 8T_{n_3} = aq^2, \quad 8T_{n_4} = aq^3.$$ 

Let $m_i = 2n_i + 1$ ($i = 1, 2, 3, 4$). Then

$$a + 1 = m_1^2, \quad aq + 1 = m_2^2, \quad aq^2 + 1 = m_3^2, \quad aq^3 + 1 = m_4^2. \quad (1)$$

Let

$$q = \frac{b_1}{a_1}, \quad a_1, b_1 \in \mathbb{Z}, \quad (a_1, b_1) = 1, \quad a_1 \geq 1.$$ 

Because $aq^3$ is positive integer, we have $a_1^3 | ab_1^3$. Noting that $(a_1, b_1) = 1$, we have $a_1^3 | a$. Let $a = a_1^2a_0, \quad a_0 \in \mathbb{N}$. By (1) we have

$$m_1^2 - a_1^3a_0 = 1, \quad m_2^2 - b_1^2a_1a_0 = 1. \quad (2)$$

Because $a = m_1^2 - 1$ and $a = a_1^3a_0 \in \mathbb{N}$, we have $a_1a_0$ is not a perfect square.

Let $x_0 + y_0\sqrt{a_0a_1}$ be the basic solution of Pell equation $x^2 - a_0a_1y^2 = 1$. Then by (2) and the theory of Pell equations, we have

$$m_1 + a_1\sqrt{a_0a_1} = (x_0 + y_0\sqrt{a_0a_1})^k, \quad m_3 + b_1\sqrt{a_0a_1} = (x_0 + y_0\sqrt{a_0a_1})^l,$$

where $k, l$ are all positive integers. By $0 < q < 1$ and (1) we have $m_1 > m_3$ and $a_1 > b_1$. So $k > l \geq 1$.

If $k = 2$, then $m_1 + a_1\sqrt{a_0a_1} = (x_0 + y_0\sqrt{a_0a_1})^2$. Thus we have $a_1 = 2x_0y_0$. So $x_0 | a_1$. Since $x_0^2 - a_0a_1y_0^2 = 1$, we have $x_0 = 1$, a contradiction with $x_0 + y_0\sqrt{a_0a_1}$ being the basic solution of Pell equation $x^2 - a_0a_1y^2 = 1$. If $k \geq 3$, then $m_1 + a_1\sqrt{a_0a_1} = (x_0 + y_0\sqrt{a_0a_1})^3$. Thus $a_1 > \binom{k}{3}x_0^{k-3}a_0a_1y_0^2$, which is obviously impossible. \qed

References