GAPS IN THE SPECTRUM OF NATHANSON HEIGHTS OF PROJECTIVE POINTS

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Abstract

Let \( \mathbb{Z}_m \) be the ring of integers modulo \( m \) (not necessarily prime), \( \mathbb{Z}_m^* \) its multiplicative group, and let \( x \mod m \) be the least nonnegative residue of \( x \) modulo \( m \). The Nathanson height of a point \( r = \langle r_1, \ldots, r_d \rangle \in (\mathbb{Z}_m^*)^d \) is \( h_m(r) = \min \left\{ \sum_{i=1}^{d} (kr_i \mod m) : k = 1, \ldots, p - 1 \right\} \). For \( d = 2 \), we give an explicit formula in terms of the convergents to the continued fraction expansion of \( r_1r_2/m \). Further, we show that the multiset \( \{ m^{-1}h_m((r_1, r_2)) : m \in \mathbb{N}, r_i \in \mathbb{Z}_m^* \} \), which is trivially a subset of \([0, 2] \), has only the numbers \( 1/k \) \( (k \in \mathbb{Z}^+) \) and \( 0 \) as accumulation points.

1. Introduction

In [3], Nathanson and Sullivan raised the problem of bounding the height of points in \( (\mathbb{Z}_p^*)^d \), where \( p \) is a prime. After proving some general bounds for \( d > 2 \), they move to identifying those primes \( p \) and residues \( r \) with \( h_p((1, r)) > (p - 1)/2 \). In particular, they prove that if \( h_p((1, r)) < p \), then it is in fact at most \( (p + 1)/2 \). Nathanson has further proven [2] that if \( p \) is a sufficiently large prime and \( h_p((1, r)) < (p + 1)/2 \), then it is in fact at most \( (p + 4)/3 \). In other words, \( p^{-1}h_p((1, r)) \) is either near 1, near 1/2, or at most 1/3.

In this paper we show that these gaps in the values of \( p^{-1}h_p((1, r)) \) continue all the way to 0, even if \( p \) is not restricted to be prime. The main tool is the simple continued fraction of \( r/p \).

To avoid confusion, as we do not use primeness here, and since the numerators of continued fractions are traditionally denoted by \( p \), we denote our modulus by \( m \). We denote \( a^{-1} \mod m \) by \( \bar{a} \). We use the traditional notation for the floor function (\( \lfloor x \rfloor \) is the largest integer that isn’t larger than \( x \)) and the fractional part (\( \{ x \} = x - \lfloor x \rfloor \)).

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If \( \gcd(r_1, m) = 1 \), then \( h_m((r_1, r_2)) = h_m((1, r_1 r_2)) \), and so we may assume without loss of generality that \( r_1 = 1 \). We are thus justified in making the following definition for relatively prime positive integers \( r, m \):

\[
H(r/m) := m^{-1} \cdot h_m((1, r)) = m^{-1} \cdot \min\{k + (kr \mod m) : 1 \leq k < m\} = \min \left\{ \frac{k}{m} + \left\{ \frac{kr}{m} \right\} : 1 \leq k < m \right\}.
\]

Figure 1 shows the points \((\frac{r}{m}, H(\frac{r}{m}))\) for all \( r, m \leq 200 \).

The spectrum of a set \( M \subseteq \mathbb{N} \), written \( \text{Spec}(M) \), is the set of real numbers \( \beta \) with the property that there are \( m_i \in M \), \( m_i \to \infty \), and a sequence \( r_i \) with \( \gcd(r_i, m_i) = 1 \), and \( H(r_i/m_i) \to \beta \). Nathanson [2] and Nathanson and Sullivan [3] proved that

\[
\text{Spec(primes)} \cap \left[ \frac{1}{3}, \infty \right) = \left\{ \frac{1}{3}, \frac{1}{2}, 1 \right\}.
\]

Our main theorem concerns the spectrum of Nathanson heights, and applies to both \( \mathbb{N} \) and to the set of primes.

**Theorem 1.1.** Let \( M \subseteq \mathbb{Z}^+ \). If \( \{m \in M : \gcd(m, n) = 1\} \) is infinite for every positive integer \( n \), then

\[
\text{Spec}(M) = \{0\} \cup \left\{ \frac{1}{k} : k \in \mathbb{Z}^+ \right\}.
\]
2. Continued Fractions

For a rational number $0 < r/m < 1$, let $[0; a_1, a_2, \ldots, a_n]$ be (either one of) its simple continued fraction expansion, and let $p_k/q_k$ be the $k$-th convergent. In particular

$$
\frac{p_0}{q_0} = \frac{0}{1},
\frac{p_2}{q_2} = \frac{a_2}{1 + a_1 a_2},
\frac{p_4}{q_4} = \frac{a_2 + a_4 + a_2 a_3 a_4}{1 + a_1 a_2 + a_1 a_4 + a_3 a_4 + a_1 a_2 a_3 a_4}
$$

The $q_i$ satisfy the recurrence $q_{-2} = 1, q_{-1} = 0, q_n = a_n q_{n-1} + q_{n-2}$ (with $a_0 = 0$), and are called the continuants. The intermediants are the numbers $\alpha q_{n-1} + q_{n-2}$, where $\alpha$ is an integer with $1 \leq \alpha \leq a_n$.

Let $E[a_0, a_1, \ldots, a_n]$ be the denominator $[a_0; a_1, \ldots, a_n]$, considered as a polynomial in $a_0, \ldots, a_n$, and set $E[\ ] = 1$. Then $p_k = E[a_0, \ldots, a_k]$ and $q_k = E[a_1, \ldots, a_k]$. We will make use of the following combinatorial identities, which are in [4, Chapter 13], with $0 < s < t < n$:

$$
q_\ell = q_k E[a_{k+1}, \ldots, a_\ell] + q_{k-1} E[a_k + 2, \ldots, a_\ell],
$$

$$
p_n E[a_s, \ldots, a_\ell] - p_\ell E[a_s, \ldots, a_n] = (-1)^{t-s+1} E[a_0, \ldots, a_{s-2}] E[a_{t+2}, \ldots, a_n].
$$

The following lemmas are well known. The first is a special case of the “best approximations theorem” [1, Theorems 154 and 182], and the second is an application of [1, Theorem 150], the identity $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$. The third and fourth lemmas follow from the identities for $E$ given above.

**Lemma 2.1.** Fix a real number $x = [0; a_1, a_2, \ldots]$, and suppose that the positive integer $\ell$ has the property that $\ell x \leq k x$ for all positive integers $k \leq \ell$. Then there are nonnegative integers $n, \alpha \leq a_n$ such that $\ell = \alpha q_{2n-1} + q_{2n-2}$.

**Lemma 2.2.** Let $\frac{p_{2k}}{q_{2k}} = [0; a_1, a_2, \ldots, a_{2k}]$, and let $x = [0; a_1, a_2, \ldots, a_{2k-1}, a_{2k} + 1]$. Then

$$
q_{2k} \cdot x - p_{2k} = \frac{1}{2q_{2k} + q_{2k-1}}.
$$

We will use Fibonacci numbers, although the only property we will make use of is that they tend to infinity: $F_1 = 1, F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$.

**Lemma 2.3.** For all $k \geq 1$, we have $q_k \geq F_k$. Further, for $\ell > k$, we have

$$
q_\ell > q_k F_{\ell-k}, \quad \text{and} \quad q_\ell > a_\ell q_k.
$$
Lemma 2.4. For $0 < 2k + 2 \leq n$, we have
\[ q_{2k}p_n - p_{2k}q_n = E[a_{2k+2}, \ldots, a_n]. \]
Moreover, if $2k + 2 = n + 1$, then $q_{2k}p_n - p_{2k}q_n = 1$.

We now state and prove our formula for heights.

Theorem 2.5. Let $\frac{r}{m} = [0; a_1, a_2, \ldots, a_n]$ (with gcd$(r, m) = 1$). Then
\[ H\left(\frac{r}{m}\right) = \min_{0 \leq k < n/2} \left\{ \frac{q_{2k} r + 1}{m} - p_{2k} \right\}. \]

Proof. First, recall that
\[ H(r/m) = \min \{k/m + \{kr/m\} : 1 \leq k < m\}. \]
Set
\[ I := \{\alpha q_{2i-1} + q_{2i-2} : 0 \leq \alpha \leq a_{2i}, 0 \leq i \leq n/2\}. \]

We call $\ell$ a best multiplier if
\[ \frac{k}{m} + \{\ell r/m\} < k/m + \{kr/m\} \]
for all positive integers $k < \ell$. We begin by proving by induction that the set of best multipliers is contained in the set $I$. Certainly 1 is a best multiplier and also $1 = 0 \cdot q_{-1} + q_{-2} \in I$. Our induction hypothesis is that the best multipliers that are less than $\ell$ are all contained in $I$.

Suppose that $\ell$ is a best multiplier: we know that
\[ \frac{k}{m} + \{\frac{k}{m}\} > \frac{\ell}{m} + \{\frac{\ell}{m}\} \]
for all $1 \leq k < \ell$. Since $k < \ell$, we then know that $\{kr/m\} > (\ell - k)/m + \{\ell r/m\} > \{\ell r/m\}$. Lemma 2.1 now tells us that $\ell \in I$. This confirms the induction hypothesis, and establishes that
\[ H(r/m) = \min\{k/m + \{kr/m\} : k \in I\}. \]  
(1)

Now, note that the function $f_i$ defined by
\[ f_i(x) := \frac{xq_{2i-1} + q_{2i-2}}{m} + \left\{ \frac{xq_{2i-1} + q_{2i-2}}{m} \right\} \]
is monotone on the domain $0 \leq x \leq a_{2i}$. As $0q_{2i-1} + q_{2i-2} = q_{2i-2}$ and $a_{2i}q_{2i-1} + q_{2i-2} = q_{2i}$, this means that the minimum in Eq. (1) can only occur at $q_{2i}$, with $0 \leq 2i \leq n$.

As a final observation, we note that $q_0/m + \{q_0r/m\} = (r + 1)/m$ is at most as large as $q_n/m + \{q_nr/m\} = 1$ (as $q_n = m$). Thus, the minimum in Eq. (1) cannot occur exclusively at $k = q_n = m$. □
**Corollary 2.6.** Let \( 0 < r < m \), with \( \gcd(r, m) = 1 \), and let \( \frac{r}{m} = [0; a_1, \ldots, a_n] \), with \( a_n \geq 2 \). For all \( k \in (0, n/2) \),

\[
H\left( \frac{r}{m} \right) \leq \frac{q_{2k}}{m} + \frac{1}{2q_{2k}}.
\]

**Proof.** First, note that \( \frac{r}{m} < [0; a_1, a_2, \ldots, a_{2k-1}, a_{2k} + 1] \). Now, as a matter of algebra (using Lemma 2.2),

\[
q_{2k} \frac{r + 1}{m} - p_{2k} \leq q_{2k} \left( [0; a_1, a_2, \ldots, a_{2k} + 1] + \frac{1}{m} \right) - p_{2k} = \frac{q_{2k}}{m} + \frac{1}{2q_{2k} + q_{2k-1}} \\
\leq \frac{q_{2k}}{m} + \frac{1}{2q_{2k}}.
\]

\( \square \)

### 3. Proof of Theorem 1.1

First, we note that \( H(a_2/(1 + a_1a_2)) = (1 + a_2)/(1 + a_1a_2) \to 1/a_1 \), where \( a_1 \) is fixed and \( a_2 \to \infty \). Thus, \( 1/k \in \text{Spec}(\mathbb{N}) \) for every \( k \). Also, \( H(1/a_1) = 2/a_1 \to 0 \) as \( a_1 \to \infty \), so \( 0 \in \text{Spec}(\mathbb{N}) \). The remainder of this section is devoted to proving that if \( \beta > 0 \) is in \( \text{Spec}(\mathbb{N}) \), then \( \beta \) is rational with numerator 1.

Fix a large integer \( s \). Let \( r/m \) be a sequence (we will suppress the index) with \( \gcd(r, m) = 1 \) and with \( H(r/m) \to \beta > \frac{1}{F_{2s}} \), where \( F_{2s} \) is the 2s-th Fibonacci number.

Define \( a_1, a_2, \ldots \) by

\[
\frac{r}{m} = [0; a_1, a_2, \ldots, a_n],
\]

and we again remind the reader that \( r/m \) is a sequence, so that each of \( a_1, a_2, \ldots \), is a sequence, and \( n \) is also a sequence. To ease the psychological burden of considering sequences that might not even be defined for every index, we take this occasion to pass to a subsequence of \( r/m \) that has \( n \) nondecreasing. Further, we also pass to a subsequence on which each of the sequences \( a_i \) is either constant or monotone increasing.

First, we show that \( n \) is bounded. Note that \( q_{2s}/m \) (fixed \( s \)) is the same as \( q_{2s}/q_n \), and by Lemma 2.3 this is at most \( 1/(2F_{2s}) \), provided that \( n \) is large enough so that \( F_{n-2s} > 2F_{2s} \). Take such an \( n \). We have from Corollary 2.6 that

\[
H\left( \frac{r}{m} \right) \leq \frac{q_{2s}}{m} + \frac{1}{2q_{2s}} < \frac{1}{2F_{2s}} + \frac{1}{2F_{2s}} < \frac{1}{F_{2s}} < \beta.
\]

This contradicts the hypothesis that \( H(r/m) \to \beta > 0 \), and proves that \( n \) must be small enough so that \( F_{n-2s} > 2F_{2s} \).
Since \( m \to \infty \) but \( n \) is bounded, some \( a_i \) must be unbounded. Let \( i \) be the least natural number such that \( a_i \) is unbounded.

First, we show that \( i \) is not odd. If \( i = 2k + 1 \), then

\[
H\left( \frac{r}{m} \right) \leq q_{2k} \frac{r + 1}{m} - p_{2k}
\]

and \( p_{2k} \) and \( q_{2k} \) are constant. Since \( a_{2k+1} \to \infty \), the ratio

\[
\frac{r}{m} \to [0; a_1, a_2, \ldots, a_{2k}] = \frac{p_{2k}}{q_{2k}}.
\]

Thus, since \( q_{2k}/m \leq 1/a_{2k+1} \to 0 \),

\[
H\left( \frac{r}{m} \right) \leq q_{2k} \frac{r + 1}{m} - p_{2k} = q_{2k} \frac{r}{m} + q_{2k} \frac{r}{m} - p_{2k} \to q_{2k} \frac{p_{2k}}{q_{2k}} + 0 - p_{2k} = 0,
\]

contradicting the hypothesis that \( b > 0 \).

Now we show that there are not two \( a_i \)’s that are unbounded. Suppose that \( a_{2k} \) and \( a_j \) are both unbounded, with \( j > 2k \). Then

\[
H\left( \frac{r}{m} \right) \leq q_{2k} \frac{r}{m} + \frac{1}{2q_{2k}}.
\]

Since \( a_{2k} \) is unbounded, \( \frac{1}{2q_{2k}} \to 0 \). And since \( a_j \) is also unbounded,

\[
\frac{q_{2k}}{m} \leq \frac{q_{2k}}{q_j} < \frac{q_{2k}}{q_{j-1}} \cdot \frac{q_{j-1}}{q_j} < \frac{1}{F_{j-1-2k}} \cdot \frac{1}{a_j} \to 0.
\]

Thus

\[
\frac{q_{2k}}{m} + \frac{1}{2q_{2k}} \to 0.
\]

We have shown that there is exactly one \( a_i \) that is unbounded, and that \( i \) is even.

We have \( \frac{r}{m} = [0; a_1, \ldots, a_{2k}, \ldots, a_n] \), with all of the \( a_i \) fixed except \( a_{2k} \), and \( a_{2k} \to \infty \). Now

\[
\lim H\left( \frac{r}{m} \right) = \lim \min_{0 \leq j < n/2} q_{2j} \frac{r + 1}{m} - p_{2j} = \lim \min_{0 \leq j < n/2} \left( \frac{q_{2j}p_n - p_{2j}q_n + q_{2j}}{q_n} \right) = \min_{0 \leq j < n/2} \lim a_{2j} \to \infty \left( \frac{E[a_{2j+2}, \ldots, a_n] + E[a_1, \ldots, a_{2j}]}{E[a_1, \ldots, a_n]} \right)
\]

Using the general identity (for \( s \leq \ell \leq t \))

\[
E[a_s, \ldots, a_t] = a_t E[a_s, \ldots, a_{t-1}] E[a_{t+1}, \ldots, a_t] + E[a_s, \ldots, a_{t-2}] E[a_t + 1, \ldots, a_t] + E[a_s, \ldots, a_{t-1}] E[a_{t+2}, \ldots, a_t]
\]
with \( \ell = 2k \), we can evaluate the limit as \( a_{2k} \to \infty \). We arrive at

\[
\beta = \lim_{m \to \infty} \frac{H(r)}{m} = \min \left\{ \min_{0 \leq j < k} \frac{E[a_{2j+2}, \ldots, a_{2k-1}]E[a_{2k+1}, \ldots, a_n]}{E[a_1, \ldots, a_{2k-1}]E[a_{2k+1}, \ldots, a_n]}, \min_{k \leq j < n/2} \frac{E[a_1, \ldots, a_{2k-1}]E[a_{2k+1}, \ldots, a_{2j}]}{E[a_1, \ldots, a_{2k-1}]E[a_{2k+1}, \ldots, a_n]} \right\}
\]

\[
= \min \left\{ \frac{1}{E[a_1, \ldots, a_{2k-1}]}, \frac{1}{E[a_{2k+1}, \ldots, a_n]} \right\}
\]

In either case, the numerator of \( \beta \) is 1, and the proof of Theorem 1.1 is concluded.

We note that we have actually proved (with a small bit of additional algebra) a quantitative version of the Theorem.

**Theorem 3.1.** Let \((r_i, m_i)\) be a sequence of pairs of positive integers with \( \gcd(r_i, m_i) = 1 \), \( m_i \to \infty \) and \( \limsup H(r_i/m_i) > 0 \). Then there is a pair of relatively prime positive integers \( a, b \), with \( a \leq b \), a positive integer \( c \), and an increasing sequence \( i_1, i_2, \ldots \) with

\[
r_{i_j} = \frac{am_{i_j} - c}{b} \quad \text{and} \quad H\left(\frac{r_{i_j}}{m_{i_j}}\right) \to \frac{1}{\max\{c, b\}}.
\]

Conversely, if \( m_i \to \infty \), and \( a \leq b \) are two relatively prime positive integers, \( c \) is a positive integer, and \( r_i = \frac{am_i - c}{b} \) is an integer relatively prime to \( m_i \), then \( H(r_i/m_i) \to \frac{1}{\max\{c, b\}} \).

In particular, if for every \( n \) there are arbitrarily large \( m \in M \) with \( \gcd(m, n) = 1 \), then \( \text{SPEC}(M) = \text{SPEC}(\mathbb{N}) \).

**References**


