A NOTE ON A CONJECTURE OF ERDŐS-TURÁN

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Abstract

Let \( \{a_n\}_{n=1}^{\infty} \) be a strictly increasing sequence of nonnegative integers. We prove that for \( F(x) = \sum_{n=1}^{\infty} x^{a_n} \) and \( F(x)^2 = \sum_{n=0}^{\infty} R(n) x^n \), the condition \( \limsup_{n \to \infty} R(n) = A \) for some positive integer \( A \) implies that \( \liminf_{n \to \infty} R(n) \leq A - 2\sqrt{A} + 1 \).

1. Introduction

Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a strictly increasing sequence of nonnegative integers. Let

\[
F(x) = \sum_{n=1}^{\infty} x^{a_n}
\]

and

\[
F(x)^2 = \sum_{n=0}^{\infty} R(n) x^n.
\]

The sequence \( \{a_n\}_{n=1}^{\infty} \) is called an additive basis of order two if \( R(n) > 0 \) for every nonnegative integer \( n \) and an asymptotic additive basis of order two if \( R(n) > 0 \) for every sufficiently large \( n \). The Erdős-Turán conjecture says that for any additive basis of order two \( \{a_n\}_{n=1}^{\infty} \) the sequence \( \{R(n)\}_{n=0}^{\infty} \) is unbounded. This conjecture can be rephrased in number theoretic language: Let \( \{a_n\}_{n=1}^{\infty} \) be a strictly increasing sequence of integers. Denote by \( R(n) \) the number of solution \( n = a_i + a_j \), i.e.,

\[
R(n) = \# \{(i, j) : n = a_i + a_j \}.
\]

Using this representation function the Erdős-Turán conjecture can be stated as follows,

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Conjecture 1 (Erdős-Turán conjecture for bases of order two) Let \( \{a_n\}_{n=1}^{\infty} \) be a strictly increasing sequence of nonnegative integers such that \( R(n) > 0 \) for every nonnegative integer \( n \). Then the sequence \( \{R(n)\}_{n=0}^{\infty} \) is unbounded.

Grekos, Haddad, Helou and Pihko [3] proved that \( \limsup_{n \to \infty} R(n) \geq 6 \) for every basis \( \{a_n\} \). Later Borwein, Choi and Chu [1] improved it to \( \limsup_{n \to \infty} R(n) \geq 8 \).

If for some strictly increasing sequence nonnegative integers \( \{a_n\}_{n=1}^{\infty} \) the representation function \( R(n) > 0 \) for every \( n \geq n_0 \) (that is \( \{a_n\}_{n=1}^{\infty} \) forms an asymptotic additive basis), then the sequence \( \{0, 1, \ldots, n_0 - 1\} \cup \{a_n\}_{n=1}^{\infty} \) forms a basis and if its representation function is denoted by \( R'(n) \) then \( R'(n) \leq R(n) + n_0 \). Therefore, we get that the above conjecture is equivalent to

Conjecture 2 (Erdős-Turán conjecture for asymptotic bases of order 2) Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a strictly increasing sequence of nonnegative integers such that \( R(n) > 0 \) for every \( n \geq n_0 \). Then the sequence \( \{R(n)\}_{n=0}^{\infty} \) is unbounded.

This second version can be formulated as:

Conjecture 3 (Erdős-Turán conjecture for bounded representation function)
Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a strictly increasing sequence of nonnegative integers and

\[
\limsup_{n \to \infty} R(n) = A
\]

for some positive integer \( A \). Then we have \( \liminf_{n \to \infty} R(n) = 0 \).

In this note we give a non-trivial upper bound for \( \liminf_{n \to \infty} R(n) \) if the sequence \( \{R(n)\}_{n=0}^{\infty} \) is bounded.

**Theorem 1** Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a strictly increasing sequence of nonnegative integers and \( \limsup_{n \to \infty} R(n) = A \) for some positive integer \( A \). Then we have

\[
\liminf_{n \to \infty} R(n) \leq A - 2\sqrt{A} + 1.
\]

2. Proof

If \( a_N > N^2 \) for some \( N \), then

\[
\# \{ n : 1 \leq n \leq N^2, \ R(n) > 0 \} \leq \binom{N}{2},
\]
and therefore
\[ \# \{ n : \ 1 \leq n \leq N^2, \ R(n) = 0 \} \geq \binom{N + 1}{2}. \]
Hence it follows that if \( a_n > n^2 \) for infinitely many integers \( n \), then \( R(n) = 0 \) for infinitely many integers \( n \). Then we have \( \liminf_{n \to \infty} R(n) = 0 \leq A - 2\sqrt{A} + 1 \), which proves the theorem.

Therefore we may assume that
\[ a_n \leq n^2 \quad \text{for } n \geq n_1. \quad (1) \]
Let us suppose that there exists a strictly increasing sequence of nonnegative integers \( \{a_n\}_{n=1}^{\infty} \) such that \( \limsup_{n \to \infty} R(n) = A \) but \( \liminf_{n \to \infty} R(n) > A - 2\sqrt{A} + 1 \). Then there exist an integer \( n_2 \) and \( 0 < \epsilon < \sqrt{A} \) for which \( A - 2\sqrt{A} + 1 + \epsilon \leq R(n) \leq A \) for \( n \geq n_2 \). Set \( C = A - \sqrt{A} + \epsilon \). By elementary calculus we have \( f(x) = \frac{(x-C)^2}{x} < 1 \) for every \( x \in [A - 2\sqrt{A} + 1 + \epsilon, A] \), and therefore there exists a \( \delta > 0 \) such that
\[ (R(n) - C)^2 \leq (1 - \delta)^2 R(n) \quad \text{for } n \geq n_2. \quad (2) \]
Let
\[ F(z) = \sum_{n=1}^{\infty} z^{a_n}. \]
Then
\[ F(z)^2 = \sum_{n=0}^{\infty} R(n) z^n. \]
Let
\[ z = (1 - \frac{1}{N})e^{2\pi i\alpha} = re^{2\pi i\alpha}, \]
where \( N \) is a large integer. We give an upper and a lower bound for the integral
\[ \int_{0}^{1} |F(z)|^2 - \sum_{n=0}^{\infty} Cz^n|d\alpha \quad (3) \]
to reach a contradiction. We get an upper bound for (3) by Cauchy’s inequality, Parseval’s formula and (2):
\[ \int_{0}^{1} |F(z)|^2 - \sum_{n=0}^{\infty} Cz^n|d\alpha = \int_{0}^{1} |\sum_{n=0}^{\infty} (R(n) - C) z^n|d\alpha \leq \left( \int_{0}^{1} |\sum_{n=0}^{\infty} (R(n) - C) z^n|^2d\alpha \right)^{1/2} = \left( \sum_{n=0}^{\infty} (R(n) - C)^2 r^{2n} \right)^{1/2} \leq \left( c_1 + (1 - \delta)^2 \sum_{n=0}^{\infty} R(n)r^{2n} \right)^{1/2} \leq c_2 + (1 - \delta) F(r^2). \quad (4) \]
Now here is the lower bound for (3). Obviously,
\[ \int_{0}^{1} |F(z)|^2 - \sum_{n=0}^{\infty} Cz^n|d\alpha \geq \int_{0}^{1} |F(z)|^2|d\alpha - \int_{0}^{1} \sum_{n=0}^{\infty} |Cz^n|d\alpha, \quad (5) \]
where by Parseval’s formula
\[ \int_0^1 |F^2(z)|d\alpha = \sum_{n=1}^{\infty} r^{2\alpha_n} = F(r^2). \]  

(6)

Moreover
\[ \int_0^1 |\sum_{n=0}^{\infty} Cz^n|d\alpha = C \int_0^1 \frac{1}{|1-z|}d\alpha = 2C \int_0^{1/2} \frac{1}{|1-z|}d\alpha. \]

Since
\[ |1-z|^2 = (1-r \cos 2\pi \alpha)^2 + (r \sin 2\pi \alpha)^2 = (1-r)^2 + 2r(1-\cos 2\pi \alpha) = (1-r)^2 + 4r \sin^2 \pi \alpha, \]
we have \(|1 - z| \geq \max\{\frac{1}{N}, \alpha\}\) for every \(0 < \alpha < \frac{1}{2}\). Hence
\[ \int_0^1 |\sum_{n=0}^{\infty} Cz^n|d\alpha \leq 2C(\int_0^{1/N} Nd\alpha) + \int_{1/N}^{1/2} \frac{1}{\alpha}d\alpha \leq c_3 \log N \]

(7)

for some \(c_3 > 0\). By (4), (6) and (7) we have
\[ F(r^2) - c_3 \log N \leq \int_0^1 |F^2(z) - \sum_{n=0}^{\infty} Cz^n|d\alpha \leq (1 - \delta)F(r^2) + c_2; \]

therefore
\[ \delta F(r^2) < c_2 + c_3 \log N, \]

(8)

but in view of (1)
\[ F(r^2) = \sum_{n=1}^{\infty} r^{2\alpha_n} \geq \sum_{n=n_1}^{\sqrt{N}} (1 - \frac{1}{N})^{2\alpha_n} > c_4 \sqrt{N} \]

for some positive \(c_4\), which is a contradiction to (8) if \(N\) is large enough. \(\Box\)

References

