ON THE FROBENIUS PROBLEM FOR GEOMETRIC SEQUENCES

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Abstract

Let $a, b, k$ be positive integers, with $\gcd(a, b) = 1$, and let $\mathcal{A}$ denote the geometric sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$. Let $\Gamma(\mathcal{A})$ denote the set of integers that are expressible as a linear combination of elements of $\mathcal{A}$ with non-negative integer coefficients. We determine $g(\mathcal{A})$ and $n(\mathcal{A})$ which denote the largest (respectively, the number of) positive integer(s) not in $\Gamma(\mathcal{A})$. We also determine the set $\mathcal{S}^*(\mathcal{A})$ of positive integers not in $\Gamma(\mathcal{A})$ which satisfy $n + \Gamma^*(\mathcal{A}) \subset \Gamma^*(\mathcal{A})$, where $\Gamma^*(\mathcal{A}) = \Gamma(\mathcal{A}) \setminus \{0\}$.

1. Introduction

For a sequence of relatively prime positive integers $A = a_1, a_2, \ldots, a_k$, let $\Gamma(A)$ denote the set of all integers of the form $\sum_{i=1}^{k} a_ix_i$ where each $x_i \geq 0$. It is well known and not difficult to show that $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ is a finite set. The Coin Exchange Problem of Frobenius is to determine the largest integer in $\Gamma^c(A)$. This is denoted by $g(A)$, and called the Frobenius number of $A$. The Frobenius number is known in the case $k = 2$ to be $g(a_1, a_2) = a_1a_2 - a_1 - a_2$, but is generally otherwise unsolved except in some special cases. A related problem is the determination of the number of integers in $\Gamma^c(A)$, which is denoted by $n(A)$ and known in the case $k = 2$ to be given by $n(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. More complete information on the Frobenius problem may be found in [3].

Ong and Ponomarenko recently determined the Frobenius number for geometric sequences in [2]. If we denote the geometric sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$ by $\mathcal{A}_k(a, b)$, and the corresponding Frobenius number by $F_k = g(\mathcal{A}_k(a, b))$, Ong & Ponomarenko proved their claim by showing that the sequence $\{G_k\}_{k \geq 1}$ satisfies a certain first order recurrence, and then using induction. The main purpose of this note is to show that both the Frobenius number $g(A)$ and $n(A)$ follow in the case of geometric sequences from an old reduction formula due to Johnson [1] and Rödseth [4]. We further determine the set $\mathcal{S}^*$, introduced in [5], in the case of geometric sequences. This gives another proof of the result for the Frobenius number since $g(A)$ is the largest integer in $\mathcal{S}^*(A)$.
2. Main Results

Throughout this section, for positive integers \(a, b, k\) with \(\gcd(a, b) = 1\), we denote by \(\mathcal{A}_k(a, b)\) the geometric sequence \(a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k\). We derive the values of both \(g(\mathcal{A}_k(a, b)) := G_k\) and \(n(\mathcal{A}_k(a, b)) := N_k\) by two methods. We first use a well-known reduction formula to derive recurrence relations for the two sequences \(\{G_k\}_{k \geq 1}\) and \(\{N_k\}_{k \geq 1}\), and then use telescoping sums to solve each recurrence. The second method to derive \(g(\mathcal{A}_k(a, b))\) consists in showing that \(S^* (\mathcal{A}_k(a, b))\) has exactly one element, which must then be \(g(\mathcal{A}_k(a, b))\). Our second proof of the result for \(n(\mathcal{A}_k(a, b))\) is indirect; we show that \(2n(\mathcal{A}_k(a, b)) - 1 = g(\mathcal{A}_k(a, b))\). We first recall the reduction formula that is central to our first derivation.

**Lemma 1.** ([1, 4]) Let \(a_1, a_2, \ldots, a_k\) be positive integers. If \(\gcd(a_2, \ldots, a_k) = d\) and \(a_j = da_j^\prime\) for each \(j > 1\), then

(a) \(g(a_1, a_2, \ldots, a_k) = d g(a_1, a_2^\prime, \ldots, a_k^\prime) + a_1(d - 1)\);

(b) \(n(a_1, a_2, \ldots, a_k) = d n(a_1, a_2^\prime, \ldots, a_k^\prime) + \frac{1}{2}(a_1 - 1)(d - 1)\).

**Theorem 1.** Let \(a, b, k\) be positive integers, with \(\gcd(a, b) = 1\). Let \(\mathcal{A}_k(a, b)\) denote the sequence \(a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k\), and let \(\sigma_k(a, b)\) denote the sum of the integers in \(\mathcal{A}_k(a, b)\). Then

(a) \(g(\mathcal{A}_k(a, b)) = \sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1})\);

(b) \(n(\mathcal{A}_k(a, b)) = \frac{1}{2} \{ \sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1}) + 1 \}\).

**Proof.**

(a) For \(k \geq 1\), by Lemma 1, with \(a_1 = a^k\) and \(d = b\), we have

\[
g(\mathcal{A}_k(a, b)) = b g(a^k, a^{k-1}, a^{k-2} b, \ldots, ab^{k-2}, b^{k-1}) + a^k(b - 1) = b g(a^{k-1}, a^{k-2} b, \ldots, ab^{k-2}, b^{k-1}) + a^k(b - 1) = b g(\mathcal{A}_{k-1}(a, b)) + a^k(b - 1).
\]

If we write \(g(\mathcal{A}_k(a, b)) := G_k\), then the sequence \(\{G_n\}_{n \geq 1}\) satisfies the first order recurrence

\[G_n = b G_{n-1} + a^n(b - 1), \quad G_1 = g(a, b) = ab - a - b.\]

Dividing both sides of the recurrence by \(b^n\), summing from \(n = 2\) to \(n = k\) and simplifying, we get

\[
\frac{G_k}{b^k} = \frac{G_1}{b} + a^2(b - 1) \frac{b^{k-1} - a^{k-1}}{b^k(b - a)},
\]

so that

\[
g(\mathcal{A}_k(a, b)) = G_k = a^2(b - 1) \frac{b^{k-1} - a^{k-1}}{b - a} + b^{k-1}(ab - a - b) = \sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1}).
\]
(b) This is similar to part (a). For $k \geq 1$, by Lemma 1, with $a_1 = a^k$ and $d = b$, we have

$$n(A_k(a, b)) = b n(a^k, a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + \frac{1}{2}(a^k - 1)(b - 1)$$

$$= b n(a^{k-1}, a^{k-2}b, \ldots, ab^{k-2}, b^{k-1}) + \frac{1}{2}(a^k - 1)(b - 1)$$

$$= b n(A_{k-1}(a, b)) + \frac{1}{2}(a^k - 1)(b - 1).$$

If we write $n(A_k(a, b)) := N_k$, then the sequence $\{N_n\}_{n \geq 1}$ satisfies the first order recurrence

$$N_n = b N_{n-1} + \frac{1}{2}(a^n - 1)(b - 1), \quad N_1 = n(a, b) = \frac{1}{2}(a - 1)(b - 1).$$

Dividing both sides of the recurrence by $b^n$, summing from $n = 2$ to $n = k$ and simplifying, we get

$$\frac{N_k}{b^k} = \frac{N_1}{b} + \frac{1}{2} a^2(b - 1) \frac{b^{k-1} - a^{k-1}}{b^{k}(b - a)} - \frac{1}{2} b^{k-1} - 1,$$

so that

$$n(A_k(a, b)) = N_k = \frac{1}{2} a^2(b - 1) \frac{b^{k-1} - a^{k-1}}{b^{k}(b - a)} - \frac{1}{2} b^{k-1} - 1$$

$$= \frac{1}{2} \left\{1 + g(A_k(a, b))\right\}$$

$$= \frac{1}{2} \left\{\sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1}) + 1\right\}. \quad \Box$$

The formulae for both $g(A_k(a, b))$ and $n(A_k(a, b))$ in Theorem 1 display a nice symmetry in the variables $a, b$. From Theorem 1 we have $n(A_k(a, b)) = \frac{1}{2} \left\{1 + g(A_k(a, b))\right\}$. If $m, n$ are integers with sum $g(A_k(a, b))$, then it is easy to see that at most one of $m, n$ can belong to $\Gamma(A_k(a, b))$. On the other hand, if for some such pair $m, n$, neither belongs to $\Gamma(A_k(a, b))$, there would be less than $\frac{1}{2} \left\{1 + g(A_k(a, b))\right\}$ integers in $\Gamma^c(A_k(a, b))$. Thus, for every pair of non-negative integers $m, n$ with sum $g(A_k(a, b))$, exactly one of $m, n$ belong to $\Gamma^c(A_k(a, b))$. We use this to derive $n(A_k(a, b))$, giving a second proof of the assertion in the second part of Theorem 1.

**Theorem 2.** Let $a, b, k$ be positive integers, with $\gcd(a, b) = 1$. Let $A_k(a, b)$ denote the sequence $a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k$, and let $\sigma_k(a, b)$ denote the sum of the integers in $A_k(a, b)$. If $m + n = g(A_k(a, b))$, then $m \in \Gamma(A_k(a, b))$ if and only if $n \notin \Gamma(A_k(a, b))$.

**Proof.** Let $m + n = g(A_k(a, b))$. If $m \in \Gamma(A_k(a, b))$, then $n \notin \Gamma(A_k(a, b))$, for otherwise $m + n = g(A_k(a, b)) \in \Gamma(A_k(a, b))$, which is impossible.

Conversely, suppose $n \notin \Gamma(A_k(a, b))$. If $n < 0$, then $m > g(A_k(a, b))$ and so $m \in \Gamma(A_k(a, b))$. We may therefore assume that $1 \leq n \leq g(A_k(a, b))$ since both 0 and any integer greater than
$g(A_k(a,b))$ belong to $\Gamma(A_k(a,b))$. Since $n + \lambda b^k \in \Gamma(a^k, a^{k-1}b, \ldots, ab^{k-1})$ for all sufficiently large integer $\lambda$ and $n \notin \Gamma(a^k, a^{k-1}b, \ldots, ab^{k-1})$, we may write $n = \sum_{i=0}^{k-1} a^{k-i}b^i x_i - b^k x_k$, where $x_i \geq 0$ for $0 \leq i \leq k - 1$ and $x_k \geq 1$. If $x_0 > b$ in this representation, by repeatedly using the identity $a^k(x_0 - b) + a^{k-1}b(x_1 + a) = a^k x_0 + a^{k-1}b x_1$ we may assume that $0 \leq x_0 < b$ while maintaining $x_1 \geq 0$. Assuming that $x_0, x_1, \ldots, x_{j-1}$ are all non-negative integers less than $b$ for some $j < k$, by repeatedly using the identity $a^{k-j}b^j(x_j - b) + a^{k-j-1}b^{j+1}(x_{j+1} + a) = a^{k-j}b^j x_j + a^{k-j-1}b^{j+1} x_{j+1}$, we may assume that $0 \leq x_j < b$ and still have $x_{j+1} \geq 0$. Thus we may write

$$n = \sum_{i=0}^{k-1} a^{k-i}b^i x_i - b^k x_k,$$

with $0 \leq x_i \leq b - 1$ for $0 \leq i \leq k - 1$, and since $n \notin \Gamma(A_k(a,b))$, also $x_k \geq 1$. Writing $g(A_k(a,b)) = (b-1) \sum_{i=0}^{k-1} a^{k-i}b^i - b^k$, we have

$$m = g(A_k(a,b)) - n = \sum_{i=0}^{k-1} (b - 1 - x_i)a^{k-i}b^i + (x_k - 1)b^k \in \Gamma(A_k(a,b)).$$

This completes the proof.

**Corollary 1.** Let $a, b, k$ be positive integers, with $\gcd(a, b) = 1$. Then

$$n(A_k(a,b)) = \frac{1}{2} \left\{ 1 + g(A_k(a,b)) \right\}.$$

**Proof.** Consider pairs $\{m,n\}$ of integers in the interval $[0,g(A_k(a,b))]$ with $m + n = g(A_k(a,b))$. By Theorem 2, exactly one integer from each such pair is in $\Gamma^c(A_k(a,b))$. This completes the proof since no integer greater than $g(A_k(a,b))$ is in $\Gamma^c(A_k(a,b))$. \qed

**Remark 1.** Let $a, b, k$ be positive integers, with $\gcd(a, b) = 1$. Then $g(A_k(a,b))$ is an odd integer.

The evaluation of $g$ given in Theorem 1 can also be derived by explicitly determining the set $S^*$, introduced in [5], since $g(a_1, a_2, \ldots, a_k)$ is the largest element in $S^*(a_1, a_2, \ldots, a_k)$. For positive and coprime integers $a_1, a_2, \ldots, a_k$, let $\Gamma(a_1, a_2, \ldots, a_k)$ denote the non-negative integers in the set $\{a_1x_1 + a_2x_2 + \cdots + a_kx_k : x_j \geq 0\}$, let $m_j$ denote the least positive integer in $\Gamma(a_1, a_2, \ldots, a_k)$ that is congruent to $j$ mod $a_1$ for $1 \leq j \leq a_1 - 1$, and let $\Gamma^*(a_1, a_2, \ldots, a_k) = \Gamma(a_1, a_2, \ldots, a_k) \setminus \{0\}$. Then

$$S^*(a_1, a_2, \ldots, a_k) := \left\{ n \notin \Gamma(a_1, \ldots, a_k) : n + \Gamma(a_1, \ldots, a_k) \subset \Gamma^*(a_1, \ldots, a_k) \right\}$$

$$\subset \left\{ m_j - a_1 : 1 \leq j \leq a_1 - 1 \right\}.$$

Moreover,

$$m_j - a_1 \in S^*(a_1, a_2, \ldots, a_k) \iff m_j + m_i > m_{j+i} \text{ for } 1 \leq i \leq a_1 - 1. \quad (1)$$

We refer to [5] for more notations and results. With the notations above, we show that $S^*(A_k(a,b)) = \{\sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1} + b^{k+1})\}$. Since $g(a_1, a_2, \ldots, a_k) \in S^*(a_1, a_2, \ldots, a_k)$, this further verifies the first result of Theorem 1.
Lemma 2. Let \(a_1, a_2, \ldots, a_k\) be positive integers with \(\gcd(a_2, \ldots, a_k) = d\). Define, \(a'_j = a_j/d\) for \(2 \leq j \leq k\). Let \(m_j\) (respectively, \(m'_j\)) denote the least positive integer in \(\Gamma(a_1, a_2, \ldots, a_k)\) (resp., in \(\Gamma(a_1, a'_2, \ldots, a'_k)\)) that is congruent to \(j\) mod \(a_1\). Then \(m_j - a_1 \in S^*(a_1, a_2, \ldots, a_k)\) if and only if \(m'_j - a_1 \in S^*(a_1, a'_2, \ldots, a'_k)\) for \(1 \leq j \leq a_1 - 1\).

Proof. Let \(A\) denote the sequence \(a_1, a_2, \ldots, a_k\) and \(A'\) the sequence \(a_1, a'_2, \ldots, a'_k\). Since each \(m_j\) and \(m'_j\) must also be representable as a non-negative linear combination of \(a_2, \ldots, a_k\) and \(a'_2, \ldots, a'_k\) respectively, it follows that \(\{m_j : 1 \leq j \leq a_1 - 1\} = \{dm'_j : 1 \leq j \leq a_1 - 1\}\). Therefore, by (1), \(m_j - a_1 \in S^*(a_1, a_2, \ldots, a_k)\) if and only if \(m_j + m_i > m_{j+i}\) for \(1 \leq i \leq a_1 - 1\) if and only if \(m'_j + m'_i > m'_{j+i}\) for \(1 \leq i \leq a_1 - 1\) if and only if \(m'_j - a_1 \in S^*(a_1, a'_2, \ldots, a'_k)\). This completes the proof.

Theorem 3. Let \(a, b, k\) be positive integers, with \(\gcd(a, b) = 1\). Let \(A_k(a, b)\) denote the sequence \(a^k, a^{k-1}b, \ldots, ab^{-1}, b^k\), and let \(\sigma_k(a, b)\) denote the sum of the integers in \(A_k(a, b)\). Then \(S^*(A_k(a, b)) = \{\sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1})\} \) for \(k \geq 1\).

Proof. We apply Lemma 2 with \(A = A_k(a, b)\) and \(a_1 = a^k\). Then \(d = b\) and \(m_j - a^k \in S^*(A_k(a, b))\) if and only if \(\frac{1}{b}m_j - a^k \in S^*(a^k, a^{k-1}b, \ldots, ab^{-1}, b^{k-1}) = S^*(A_{k-1}(a, b))\). Therefore, by Theorem 1 in [5], \(|S^*(A_k(a, b))| = |S^*(A_1(a, b))| = 1\) for each \(k > 1\). Since we have \(g(A_k(a, b)) \in S^*(A_k(a, b))\), there can be no other integer in this set.

Corollary 2. Let \(a, b, k\) be positive integers, with \(\gcd(a, b) = 1\). Then
\[
g(A_k(a, b)) = \max S^*(A_k(a, b)) = \sigma_{k+1}(a, b) - \sigma_k(a, b) - (a^{k+1} + b^{k+1}).
\]

Remark 2. The proof of Theorem 3 shows that the sequence of Frobenius numbers \(\{g(A_k(a, b))\}_{k \geq 1}\) satisfies the recurrence \(G_k = bG_{k-1} + a^k(b - 1)\) since \(g(A_k(a, b)) = m_j - a^k\) is the only element in \(S^*(A_k(a, b))\). This result coincides with the result in the first part of Theorem 1.

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References


