DIVISIBILITY PROPERTIES OF THE 5-REGULAR AND 13-REGULAR PARTITION FUNCTIONS

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Abstract

The function \( b_k(n) \) is defined as the number of partitions of \( n \) that contain no summand divisible by \( k \). In this paper we study the 2-divisibility of \( b_5(n) \) and the 2- and 3-divisibility of \( b_{13}(n) \). In particular, we give exact criteria for the parity of \( b_5(2n) \) and \( b_{13}(2n) \).

1. Introduction

A partition of a positive integer \( n \) is a non-increasing sequence of positive integers whose sum is \( n \). In other words,

\[
    n = \lambda_1 + \lambda_2 + \cdots + \lambda_t
\]

with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 1 \). For instance, the partitions of 4 are
4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1.

We denote the number of partitions of \( n \) by \( p(n) \). So, as shown above, \( p(4) = 5 \). Note that \( p(n) = 0 \) if \( n \) is not a nonnegative integer, and we adopt the convention that \( p(0) = 1 \). The generating function for the partition function is then given by the infinite product

\[
\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots.
\]

Let \( k \) be a positive integer. We say that a partition is \( k \)-\textit{regular} if none of its summands is divisible by \( k \), and denote the number of \( k \)-regular partitions of \( n \) by \( b_k(n) \). For example, \( b_3(4) = 4 \) because the partition \( 3 + 1 \) has a summand divisible by 3 and therefore is not \( 3 \)-regular. Adopting the convention that \( b_k(0) = 1 \), the generating function for the \( k \)-regular partition function is then

\[
\sum_{n=0}^{\infty} b_k(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = \prod_{n=1}^{\infty} \frac{1 - q^{kn}}{(1 - q^n)^{k-1}}.
\]  \hspace{1cm} (1)

Note that \( b_2(n) \) equals the number of partitions of \( n \) into odd parts, which Euler proved is equal to the number of partitions of \( n \) into distinct parts.

The partition function satisfies the famous Ramanujan congruences

\[
\begin{align*}
p(5n + 4) & \equiv 0 \pmod{5}, \\
p(7n + 5) & \equiv 0 \pmod{7}, \\
p(11n + 6) & \equiv 0 \pmod{11}
\end{align*}
\]

for every \( n \geq 0 \). Ono [7] proved that such congruences for \( p(n) \) exist modulo every prime \( \geq 5 \), and Ahlgren [1] extended this to include every modulus coprime to 6. Given these facts, for a positive integer \( m \) it is natural to wonder for which values of \( n \) we have that \( p(n) \) is divisible by \( m \), or simply how often \( p(n) \) is divisible by \( m \). By the results cited above,

\[
\liminf_{X \to \infty} \frac{\#\{1 \leq n \leq X \mid p(n) \equiv 0 \pmod{m}\}}{X} > 0
\]

for any \( m \) coprime to 6. The \( m = 2 \) and \( m = 3 \) cases, meanwhile, have proven elusive.

The state of knowledge for \( k \)-regular partition functions is better. For example, Gordon and Ono [4] have shown that if \( p \) is prime, \( p^\nu \parallel k \) and \( p^\nu \geq \sqrt{k} \), then for any \( j \geq 1 \) the arithmetic density of positive integers \( n \) such that \( b_k(n) \) is divisible by \( p^j \) is one. In certain cases one can find even more specific information. As an illustration we consider the parity of \( b_2(n) \). Noting that

\[
\sum_{n=0}^{\infty} b_2(n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^n)} = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^n)} = \prod_{n=1}^{\infty} (1 - q^n) \pmod{2}
\]
by Euler’s Pentagonal Number Theorem it follows that
\[ \sum_{n=0}^{\infty} b_2(n)q^n \equiv \sum_{\ell=-\infty}^{\infty} q^{\ell(3\ell+1)/2} \pmod{2}, \]
and so \( b_2(n) \) is odd if and only if \( n = \ell(3\ell + 1)/2 \) for some \( \ell \in \mathbb{Z} \). Thus, in contrast to
the case of \( p(n) \) we have a complete answer for the 2-divisibility of \( b_2(n) \) (see [6] and [3] for
analogous results for the \( k \)-divisibility of \( b_k(n) \) for \( k \in \{3, 5, 7, 11\} \)).

Now consider the \( m \)-divisibility of \( b_k(n) \) when \( (m, k) = 1 \). In [2] Ahlgren and Lovejoy
prove that if \( p \geq 5 \) is prime, then for any \( j \geq 1 \) the arithmetic density of positive integers \( n \)
such that \( b_2(n) \equiv 0 \pmod{p^j} \) is at least \( \frac{p^{j+1}}{2p} \) (they also prove that \( b_2(n) \) satisfies Ramanujan-
type congruences modulo \( p^j \)). In [9] Penniston extended this to show that for distinct primes
\( k \) and \( p \) with \( 3 \leq k \leq 23 \) and \( p \geq 5 \), the arithmetic density of positive integers \( n \) for which
\( b_k(n) \equiv 0 \pmod{p^j} \) is at least \( \frac{p^{j+1}}{2p} \) if \( p \nmid k - 1 \), and at least \( \frac{p^{j+1}}{p} \) if \( p \mid k - 1 \) (in [11] and
[12] Treneer has shown that divisibility and congruence results such as these hold for general \( k \)). The latter result indicates that a special role may be played by the prime divisors of \( k - 1 \), and we consider this here. Upon numerically investigating the \( m \)-divisibility of \( b_k(n) \) for small values of \( k \) and \( m \) not covered by the results above, the most striking and regular patterns we found occurred for \( k = 5, m = 2 \) and for \( k = 13 \) and \( m \in \{2, 3\} \).

**Theorem 1.** Let \( n \) be a nonnegative integer. Then \( b_5(2n) \) is odd if and only if \( n = \ell(3\ell + 1) \)
for some \( \ell \in \mathbb{Z} \). That is,
\[ \sum_{n=0}^{\infty} b_5(2n)q^{2n} \equiv \sum_{\ell=-\infty}^{\infty} q^{2\ell(3\ell+1)} \pmod{2}. \]

**Remark.** By Euler’s Pentagonal Number Theorem, Theorem 1 is equivalent to
\[ \sum_{n=0}^{\infty} b_5(2n)q^{2n} \equiv \prod_{n=1}^{\infty} (1 - q^n)^4 \pmod{2}. \]  

(2)

**Theorem 2.** Let \( n \) be a nonnegative integer. Then \( b_{13}(2n) \) is odd if and only if \( n = \ell(\ell + 1) \)
or \( n = 13\ell(\ell + 1) + 3 \) for some nonnegative integer \( \ell \). That is,
\[ \sum_{n=0}^{\infty} b_{13}(2n)q^{2n} \equiv \sum_{\ell=0}^{\infty} q^{2\ell(\ell+1)} + \sum_{\ell=0}^{\infty} q^{26\ell(\ell+1)+6} \pmod{2}. \]

**Remark.** Jacobi’s triple product formula yields
\[ \prod_{n=1}^{\infty} (1 - q^n)^3 = \sum_{\ell=0}^{\infty} (-1)^\ell (2\ell + 1)q^{\ell(\ell+1)/2}, \]
and hence Theorem 2 is equivalent to
\[ \sum_{n=0}^{\infty} b_{13}(2n)q^{2n} \equiv \prod_{n=1}^{\infty} (1 - q^{4n})^3 + q^6 \cdot \prod_{n=1}^{\infty} (1 - q^{52n})^3 \pmod{2}. \]

(3)
Theorems 1 and 2 yield infinitely many Ramanujan-type congruences modulo 2 for \( b_5(n) \) and \( b_{13}(n) \) in even arithmetic progressions. It turns out that our proof of Theorem 1 yields two congruences for \( b_5(n) \) in odd arithmetic progressions.

**Theorem 3.** For every \( n \geq 0 \),
\[
  b_5(20n + 5) \equiv 0 \pmod{2}
\]
and
\[
  b_5(20n + 13) \equiv 0 \pmod{2}.
\]

Finally, we make the following conjecture regarding the 3-divisibility of \( b_{13}(n) \).

**Conjecture 1.** For any \( \ell \geq 2 \),
\[
  b_{13}\left(3^\ell n + \frac{5 \cdot 3^{\ell-1} - 1}{2}\right) \equiv 0 \pmod{3}
\]
for every \( n \geq 0 \).

It turns out (see Proposition 2 below) that one can reduce the verification of each of the congruences in Conjecture 1 to a finite computation. We have verified the conjecture for each \( 2 \leq \ell \leq 6 \) (one can easily check that the conjecture does not hold for \( \ell = 1 \)).

2. Modular Forms

We begin with some background on the theory of modular forms. Given a positive integer \( N \), let
\[
  \Gamma_0(N) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.
\]
Let \( \mathbb{H} := \{ z \in \mathbb{C} \mid \Im(z) > 0 \} \) be the complex upper half plane, and for \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \) and \( z \in \mathbb{H} \) define \( \gamma z := \frac{az + b}{cz + d} \). Throughout, we let \( q := e^{2\pi i z} \).

Suppose \( k \) is a positive integer, \( f : \mathbb{H} \rightarrow \mathbb{C} \) is holomorphic and \( \chi \) is a Dirichlet character modulo \( N \). Then \( f \) is said to be a **modular form of weight \( k \) on \( \Gamma_0(N) \)** with character \( \chi \) if
\[
  f(\gamma z) = \chi(d)(cz + d)^k f(z)
\]
for all \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) \) and \( f \) is holomorphic at the cusps of \( \Gamma_0(N) \). The modular forms of weight \( k \) on \( \Gamma_0(N) \) with character \( \chi \) form a finite-dimensional complex vector space which we denote by \( M_k(\Gamma_0(N), \chi) \) (we will omit \( \chi \) from our notation when it is the trivial character). For instance, if we denote by \( \theta(z) \) the classical theta function
\[
  \theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots,
\]
then $\theta^4(z) \in M_2(\Gamma_0(4))$ (see, for example, [5]).

A theorem of Sturm [10] provides a method to test whether two modular forms are congruent modulo a prime. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ has integer coefficients and $m$ is a positive integer, let $\text{ord}_m(f(z))$ be the smallest $n$ for which $a(n) \not\equiv 0 \pmod{m}$ (if there is no such $n$, we define $\text{ord}_m(f(z)) := \infty$).

**Theorem 4.** (Sturm) Suppose $p$ is prime and $f(z), g(z) \in M_k(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$. If

$$\text{ord}_p(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_0(N)],$$

then $f(z) \equiv g(z) \pmod{p}$, i.e., $\text{ord}_p(f(z) - g(z)) = \infty$.

We note here that $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \cdot \prod (\ell + 1)$, where the product is over the prime divisors of $N$.

Hecke operators play a crucial role in the proofs of our results. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathbb{Z}[[q]]$ and $p$ is prime, then the action of the Hecke operator $T_{p,k,\chi}$ on $f(z)$ is defined by

$$(f \mid T_{p,k,\chi})(z) := \sum_{n=0}^{\infty} (a(pn) + \chi(p)p^{k-1}a(n/p))q^n$$

(we follow the convention that $a(x) = 0$ if $x \not\in \mathbb{Z}$). Notice that if $k > 1$, then

$$(f \mid T_{p,k,\chi})(z) \equiv \sum_{n=0}^{\infty} a(pn)q^n \pmod{p}. \quad (5)$$

Moreover, if $f(z) \in M_k(\Gamma_0(N), \chi)$, then $(f \mid T_{p,k,\chi})(z) \in M_k(\Gamma_0(N), \chi)$. When $k$ and $\chi$ are clear from context, we will write $T_p := T_{p,k,\chi}$.

The next proposition follows directly from (5) and the definition of $T_{p,k,\chi}$.

**Proposition 1.** Suppose $p$ is prime, $g(z) \in \mathbb{Z}[[q]]$, $h(z) \in \mathbb{Z}[[q^p]]$ and $k > 1$. Then $(gh \mid T_{p,k,\chi})(z) \equiv (g \mid T_{p,k,\chi})(z) \cdot h(z/p) \pmod{p}$.

We will construct modular forms using Dedekind’s eta function, which is defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

for $z \in \mathbb{H}$. A function of the form

$$f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z), \quad (6)$$

where $r_\delta \in \mathbb{Z}$ and the product is over the positive divisors of $N$, is called an *eta-quotient*. 


From ([8], p. 18), if \( f(z) \) is the eta-quotient (6), \( k := \frac{1}{2} \sum_{\delta \mid N} r_\delta \in \mathbb{Z} \),
\[
\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24}
\]
and
\[
N \sum_{\delta \mid N} \frac{r_\delta}{\delta} \equiv 0 \pmod{24},
\]
then \( f(z) \) satisfies the transformation property (4) for all \( \gamma \in \Gamma_0(N) \). Here \( \chi \) is given by \( \chi(d) := \left( \frac{-1}{d} \right)^s \), where \( s := \prod_{\delta \mid N} \delta^s \). Assuming that \( f \) satisfies these conditions, then since \( \eta(z) \) is analytic and does not vanish on \( \mathbb{H} \), we have that \( f(z) \in M_k(\Gamma_0(N), \chi) \) if \( f(z) \) is holomorphic at the cusps of \( \Gamma_0(N) \). By ([8], Theorem 1.65) we have that if \( c \) and \( d \) are positive integers with \((c,d) = 1 \) and \( d \mid N \), then the order of vanishing of \( f(z) \) at the cusp \( \frac{c}{d} \) is
\[
\frac{N}{24d(d, \frac{N}{d})} \sum_{\delta \mid N} \frac{(d,\delta)^2r_\delta}{\delta}.
\]

3. Proof of the Main Results

Proof of Theorem 1. We begin with the modular forms
\[
f(z) := \frac{\eta^5(5z)}{\eta(z)} = q + q^2 + 2q^3 + 3q^4 + 5q^5 + \cdots
\]
and
\[
g(z) := \eta^4(z)\eta^4(5z) = q \cdot \prod_{n=1}^{\infty} (1 - q^n)^4(1 - q^{5n})^4.
\]

Define the character \( \chi_m \) by \( \chi_m(d) := \left( \frac{m}{d} \right) \). Using the results on eta-quotients cited above we find that \( f(z) \in M_2(\Gamma_0(5), \chi_5) \) and \( g(z) \in M_4(\Gamma_0(5)) \). Next, recall that
\[
\theta^4(z) = 1 + 8q + 24q^2 + 32q^3 + \cdots \in M_2(\Gamma_0(4)).
\]
Notice that \((\theta^4(z))^2 \in M_4(\Gamma_0(20))\).

From (1) we have
\[
f(z) = \frac{\eta(5z)}{\eta(z)} \cdot \eta^4(5z)
\]
\[
= \frac{q^{5/24} \prod_{n=1}^{\infty} (1 - q^{5n})}{q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)} \cdot q^{20/24} \prod_{j=1}^{\infty} (1 - q^{5j})^4
\]
\[
= \sum_{n=0}^{\infty} b_5(n)q^{n+1} \cdot \prod_{j=1}^{\infty} (1 - q^{20j}) \pmod{2}.
\]
It follows from Proposition 1 that

\[(f \mid T_2)(z) \equiv \sum_{n=0}^{\infty} b_5(2n + 1)q^{n+1} \cdot \prod_{j=1}^{\infty} (1 - q^{10j}) \pmod{2}, \tag{10}\]

and hence

\[h(z) := f(z) - (f \mid T_2)(2z) \equiv \sum_{n=0}^{\infty} b_5(2n)q^{2n+1} \cdot \prod_{j=1}^{\infty} (1 - q^{20j}) \pmod{2}. \tag{11}\]

Note that \(f(z)\) and \((f \mid T_2)(2z)\) are in \(M_2(\Gamma_0(10), \chi_5)\), and hence \(h(z)\) lies in this space as well. It follows that \(h^2(z)\theta^8(z) \in M_8(\Gamma_0(20))\). Now, \(g^2(z) \in M_8(\Gamma_0(20))\), and one can check that the forms \(h^2(z)\theta^8(z)\) and \(g^2(z)\) are congruent modulo 2 out to their \(q^{24}\) terms. By Sturm's theorem we conclude that these forms are congruent modulo 2. Since \(\theta(z) \equiv 1 \pmod{2}\), we have that \(h^2(z) \equiv g^2(z) \pmod{2}\), and hence \(h(z) \equiv g(z) \pmod{2}\). Then by (11) and (7),

\[\sum_{n=0}^{\infty} b_5(2n)q^{2n} \cdot \prod_{j=1}^{\infty} (1 - q^{20j}) \equiv \prod_{n=1}^{\infty} (1 - q^n)^4(1 - q^{5n})^4 \pmod{2}. \tag{12}\]

Since \((1 - q^{5n})^4 \equiv 1 - q^{20n} \pmod{2}\), (2) now follows from (12).

\[\square\]

**Proof of Theorem 2.** To begin, we define

\[u(z) := \frac{\eta^{13}(13z)}{\eta(z)} \in M_6(\Gamma_0(13), \chi_{13}).\]

We will also use the following two forms in \(M_{12}(\Gamma_0(13)):\)

\[v(z) := \eta^{12}(z)\eta^{12}(13z) = q^7 \cdot \prod_{n=1}^{\infty} (1 - q^n)^{12}(1 - q^{13n})^{12}\tag{13}\]

and

\[w(z) := \eta^{24}(13z) = q^{13} \cdot \prod_{n=1}^{\infty} (1 - q^{13n})^{24}.\tag{14}\]

From (1) we have that

\[u(z) \equiv \sum_{n=0}^{\infty} b_{13}(n)q^{n+7} \cdot \prod_{j=1}^{\infty} (1 - q^{52j})^3 \pmod{2}.\]

Then

\[(u \mid T_2)(z) \equiv \sum_{n=0}^{\infty} b_{13}(2n + 1)q^{n+4} \cdot \prod_{j=1}^{\infty} (1 - q^{26j})^3 \pmod{2},\]
and hence
\[ m(z) := u(z) - (u \mid T_2)(2z) = \sum_{n=0}^{\infty} b_{13}(2n)q^{2n+7} \cdot \prod_{j=1}^{\infty} (1 - q^{52j})^3 \quad (\text{mod } 2). \] (15)

Note that since \( u(z) \) and \( (u \mid T_2)(2z) \) lie in \( M_6(\Gamma_0(26), \chi_{13}) \), so does \( m(z) \). Then since \( \theta^{24}(z) \in M_{12}(\Gamma_0(52)) \), we have that \( m^2(z)\theta^{24}(z) \in M_{24}(\Gamma_0(52)) \). Note that \( v^2(z), w^2(z) \in M_{24}(\Gamma_0(52)) \) as well, and one can check that the forms \( m^2(z)\theta^{24}(z) \) and \( v^2(z) + w^2(z) \) are congruent modulo 2 out to their \( q^{168} \) terms. By Sturm’s theorem we conclude that
\[ m^2(z)\theta^{24}(z) \equiv v^2(z) + w^2(z) \quad (\text{mod } 2), \]
and therefore \( m(z)\theta^{12}(z) \equiv v(z) + w(z) \quad (\text{mod } 2) \). Since \( \theta(z) \equiv 1 \quad (\text{mod } 2) \), we find that \( m(z) \equiv v(z) + w(z) \quad (\text{mod } 2) \). Then (15), (13) and (14) give
\[ \sum_{n=0}^{\infty} b_{13}(2n)q^{2n+7} \cdot \prod_{j=1}^{\infty} (1 - q^{13j})^{12} \equiv q^7 \cdot q \prod_{n=1}^{\infty} (1 - q^n)^{12} (1 - q^{13n})^{12} \]
\[ + q^{13} \prod_{n=1}^{\infty} (1 - q^{13n})^{24} \quad (\text{mod } 2), \]
which implies (3).

**Proof of Theorem 3.** We prove only the first congruence, as the second can be proved in a similar way. Sturm’s theorem gives that \( f(z) \) and \( (f \mid T_2)(z) \) are congruent modulo 2, which by (10) yields
\[ \sum_{n=0}^{\infty} b_5(2n + 1)q^{n+1} \cdot \prod_{j=1}^{\infty} (1 - q^{10j}) \equiv q \prod_{n=1}^{\infty} \frac{1 - q^{5n}}{(1 - q^n)} \quad (\text{mod } 2). \]

Then
\[ \sum_{n=0}^{\infty} b_5(2n + 1)q^{n+1} \cdot \prod_{j=1}^{\infty} (1 - q^{10j}) \equiv q \prod_{n=1}^{\infty} \frac{1 - q^{5n}}{(1 - q^n)} \cdot \prod_{j=1}^{\infty} (1 - q^{5j})^4 \quad (\text{mod } 2), \]
and hence
\[ \sum_{n=0}^{\infty} b_5(2n + 1)q^n \equiv \sum_{\ell=0}^{\infty} b_5(\ell)q^{\ell} \cdot \prod_{j=1}^{\infty} (1 - q^{10j}) \quad (\text{mod } 2). \] (16)

Note that \( 2n + 1 \) has the form \( 20m + 5 \) if and only if \( n \equiv 2 \quad (\text{mod } 10) \). Since the infinite product on the right hand side of (16) only produces powers of \( q \) that are 0 modulo 10, it suffices to show that
\[ b_5(10n + 2) \equiv 0 \quad (\text{mod } 2) \] (17)
for all \( n \geq 0 \). One can easily check that the congruence \( 6\ell^2 + 2\ell \equiv 2 \quad (\text{mod } 10) \) has no solution, and so (17) follows from Theorem 1.
With regard to Conjecture 1, we have the following elementary proposition.

**Proposition 2.** Let $\ell \geq 2$. If the congruence

$$b_{13} \left( 3^{\ell} n + \frac{5 \cdot 3^{\ell-1} - 1}{2} \right) \equiv 0 \pmod{3}$$

holds for all $0 \leq n \leq 7 \cdot 3^{\ell-1} - 3$, then it holds for all $n \geq 0$.

**Proof.** The idea of our proof is to repeatedly apply the $T_3$ operator to the modular form

$$P_\ell(z) := \frac{\eta(13z)}{\eta(z)} \cdot \eta^{\ell}(13z),$$

where $e := 4 \cdot 3^\ell$. By the criteria for eta-quotients cited above, $P_\ell(z) \in M_4(\Gamma_0(13), \chi_{13})$.

For each $1 \leq t \leq \ell$ let

$$\delta_t := \frac{13 \cdot 3^{\ell-1} + 1}{2}.$$

Then

$$P_\ell(z) = \sum_{n=0}^{\infty} b_{13}(n) q^{n+\delta_t} \cdot \prod_{j=1}^{\infty} (1 - q^{13j})^e.$$

Note that

$$P_\ell(z) \equiv \sum_{n=0}^{\infty} b_{13}(n) q^{n+\delta_t} \cdot \prod_{j=1}^{\infty} (1 - q^{3^{\ell-1} \cdot 13j})^4 \pmod{3}.$$

Using Proposition 1 and the fact that $\delta_t \equiv 2 \pmod{3}$ for $2 \leq t \leq \ell$, an easy induction argument gives that $(P_\ell \mid T_3^{s-1})(z)$ is congruent modulo 3 to

$$\sum_{n=0}^{\infty} b_{13} \left( 3^{s-1} n + \left( \frac{3^{s-1} - 1}{2} \right) \right) q^{n+\delta_{s-1}} \cdot \prod_{j=1}^{\infty} (1 - q^{3^{\ell-1} \cdot 13j})^4$$

for any $1 \leq s \leq \ell - 1$. In particular,

$$(P_\ell \mid T_3^{\ell-1})(z) \equiv \sum_{n=0}^{\infty} b_{13} \left( 3^{\ell-1} n + \left( \frac{3^{\ell-1} - 1}{2} \right) \right) q^{n+7} \cdot \prod_{j=1}^{\infty} (1 - q^{3^{\ell-1} \cdot 13j})^4 \pmod{3}.$$

Then

$$(P_\ell \mid T_3^{\ell})(z) \equiv \sum_{n=0}^{\infty} b_{13} \left( 3^{\ell-1} (3n + 2) + \left( \frac{3^{\ell-1} - 1}{2} \right) \right) q^{\frac{(3n+2)^{\ell+7}}{3}} \cdot \prod_{j=1}^{\infty} (1 - q^{13j})^4 \pmod{3}.$$

Since $(P_\ell \mid T_3^{\ell})(z) \in M_4(\Gamma_0(13), \chi_{13})$, by Sturm’s theorem we have that if $\text{ord}_3((P_\ell \mid T_3^{\ell})(z)) > 7 \cdot 3^{\ell-1}$, then $(P_\ell \mid T_3^{\ell})(z) \equiv 0 \pmod{3}$. Therefore, if the congruence

$$b_{13} \left( 3^{\ell} n + \frac{5 \cdot 3^{\ell-1} - 1}{2} \right) \equiv 0 \pmod{3}$$

holds for all $0 \leq n \leq 7 \cdot 3^{\ell-1} - 3$, then it holds for all $n \geq 0$. \qed
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