NEITHER $\prod_{k=1}^{n}(4k^2 + 1)$ NOR $\prod_{k=1}^{n}(2k(k-1) + 1)$ IS A PERFECT SQUARE

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Abstract
In this paper, by employing Cilleruelo’s method, we prove that neither $\prod_{k=1}^{n}(4k^2 + 1)$ nor $\prod_{k=1}^{n}(2k(k-1) + 1)$ is a perfect square for all $n > 1$, which confirms a conjecture of Amdeberhan, Medina, and Moll.

1. Introduction
Recently, there has been a renewed interest in investigating whether or not certain product sequences contain perfect squares. Amdeberhan, Medina and Moll [1] proposed several conjectures in this direction. Soon after, J. Cilleruelo [2] proved that the number
$$\prod_{k=1}^{n}(k^2 + 1)$$
is not a perfect square provided $n > 3$, which settles Conjecture 5.1 in [1]. Amdeberhan, Medina and Moll [1] also proposed the following conjecture.

Conjecture 1 ([1, Conjecture 7.1]). The even and odd parts of $\prod_{k=1}^{n}(k^2 + 1)$ are defined by
$$t_n := \prod_{k=1}^{n}(1 + 2k(k-1)), \text{ and } s_n := \prod_{k=1}^{n}(1 + 4k^2).$$
These products involve the triangular and square numbers respectively. Neither of them is a perfect square.

In this paper, by employing Cilleruelo’s method, we confirm this conjecture.

Theorem 2. Neither $\prod_{k=1}^{n}(4k^2 + 1)$ nor $\prod_{k=1}^{n}(2k(k-1) + 1)$ is a perfect square for all $n > 1$.  

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2. Proof of Theorem 2

Proof. In this paper, \( p \) always denotes a rational prime.

Let \( P_n = \prod_{k=1}^{n} (4k^2 + 1) \). Assume that \( P_n \) is a perfect square for some \( n > 1 \). Let \( p \) be a prime with \( p|P_n \). Then \( p^2|P_n \) and \( p \equiv 1 \pmod{4} \). If there exists a positive integer \( k \leq n \) with \( p^2|4k^2 + 1 \), then \( p \leq \sqrt{4n^2 + 1} < 2n + 1 \). Thus \( p < 2n \). If there exist \( i, j, 1 \leq i < j \leq n \) with \( p|4i^2 + 1 \) and \( p|4j^2 + 1 \), then \( p|4(j - i)(j + i) \). Thus either \( p|j - i \) or \( p|j + i \). So \( p \leq j + i < 2n \).

Hence

\[
P_n = \prod_{p \equiv 1 \pmod{4}} p^{\alpha_p}.
\]

Let \( n! = \prod_{p \leq n} p^{\beta_p} \). Since \( 4^n n!^2 < P_n \), we have

\[
\sum_{p \leq n} \beta_p \log p < \frac{1}{2} \sum_{p \equiv 1 \pmod{4}} \alpha_p \log p - n \log 2. \tag{1}
\]

Since each interval of length \( p^j \) contains at most two solutions of \( 4x^2 + 1 \equiv 0 \pmod{p^j} \), we have

\[
\alpha_p = \sum_{j \leq \log (4n^2 + 1)/\log p} \#\{k \leq n : p^j|4k^2 + 1\} \leq \sum_{j \leq \log (4n^2 + 1)/\log p} 2\lfloor n/p^j \rfloor. \tag{2}
\]

On the other hand

\[
\beta_p = \sum_{j \leq \log n/\log p} \#\{k \leq n : p^j|k\} = \sum_{j \leq \log n/\log p} \lfloor n/p^j \rfloor. \tag{3}
\]

Thus we have

\[
\frac{\alpha_p}{2} - \beta_p \leq \sum_{j \leq \log (4n^2 + 1)/\log p} \lfloor n/p^j \rfloor - \sum_{j \leq \log n/\log p} \lfloor n/p^j \rfloor
\]

\[
= \sum_{j \leq \log n/\log p} \left( \lfloor n/p^j \rfloor - \lfloor n/p^j \rfloor \right) + \sum_{\log n/\log p < j \leq \log (4n^2 + 1)/\log p} \lfloor n/p^j \rfloor
\]

\[
\leq \frac{\log (4n^2 + 1)}{\log p}. \tag{4}
\]

By (1) and (4) we have

\[
\sum_{p \not\equiv 1 \pmod{4}} \beta_p \log p = \sum_{p \leq n} \beta_p \log p - \sum_{p \equiv 1 \pmod{4}} \beta_p \log p
\]

\[
\leq \frac{1}{2} \sum_{n < p < 2n} \alpha_p \log p - n \log 2 + \log (4n^2 + 1) \pi(n; 1, 4), \tag{5}
\]
where \( \pi(n; 1, 4) \) denotes the number of primes which are less than or equal to \( n \) and congruent to 1 modulo 4.

If \( p > n \), then
\[
\frac{\log (4n^2 + 1)}{\log p} < \frac{\log (n + 1)^3}{\log (n + 1)} = 3.
\]

By (2) we have \( \alpha_p \leq 4 \).

If \( p \leq n \), then by (3) we have
\[
\beta_p \geq \sum_{j \leq \log n / \log p} \left( \frac{n}{p^j} - 1 \right) = n \left( \frac{1 - p^{-1 - \lfloor \log n / \log p \rfloor}}{1 - 1/p} - 1 \right) - \lfloor \log n / \log p \rfloor.
\]
\[
\geq n \left( \frac{1 - 1/n}{1 - 1/p} - 1 \right) - \lfloor \log n / \log p \rfloor = \frac{n - p}{p - 1} - \lfloor \log n / \log p \rfloor
\]
\[
\geq \frac{n - 1}{p - 1} \frac{\log (4n^2 + 1)}{\log p},
\]

where the last inequality is based on the fact \( p \leq n \).

Thus, by (5) we have
\[
(n - 1) \sum_{p \leq n \atop p \not\equiv 1 (\mod 4)} \frac{\log p}{p - 1} < \log (4n^2 + 1) \pi(n) + 2 \sum_{n < p < 2n} \log p - n \log 2,
\]

where \( \pi(n) \) denotes the number of primes which are less than or equal to \( n \).

Now we use the Chebyshev’s estimates
\[
\sum_{p \leq n} \log p \leq 2n \log 2, \quad \sum_{n < p < 2n} \log p \leq 2n \log 2
\]

and (see [3])
\[
\pi(x) \leq \frac{x}{\log x} \left( 1 + \frac{1.2762}{\log x} \right) \quad (x > 1)
\]

to obtain
\[
\sum_{p \leq n \atop p \not\equiv 1 (\mod 4)} \frac{\log p}{p - 1} \leq \frac{\log (4n^2 + 1)}{n - 1} \left( \frac{n}{\log n} + \frac{1.2762n}{\log^2 n} \right) + \frac{3n}{n - 1} \log 2.
\]

We know that the right-hand side is monotonic decreasing. Actually, that quantity is less than 7.14 for \( n \geq 702007 \).

For \( n \geq 702007 \), we have
\[
\sum_{p \leq n \atop p \not\equiv 1 (\mod 4)} \frac{\log p}{p - 1} \geq \sum_{n < p \leq 702007} \frac{\log p}{p - 1} > 7.14,
\]

which proves the theorem for \( n \geq 702007 \).
Finally we have to check that \( P_n \) is not a square for \( 2 \leq n < 702007 \).

- \( 17 = 4 \times 2^2 + 1 \). The next time that the prime 17 divides \( 4k^2 + 1 \) is for \( k = 17 - 2 = 15 \). Hence \( P_n \) is not a square for \( 2 \leq n \leq 14 \).

- \( 101 = 4 \times 5^2 + 1 \). The next time that the prime 101 divides \( 4k^2 + 1 \) is for \( k = 101 - 5 = 96 \). Hence \( P_n \) is not a square for \( 5 \leq n \leq 95 \).

- \( 1297 = 4 \times 18^2 + 1 \). The next time that the prime 1297 divides \( 4k^2 + 1 \) is for \( k = 1297 - 18 = 1279 \). Hence \( P_n \) is not a square for \( 18 \leq n \leq 1278 \).

- \( 739601 = 4 \times 430^2 + 1 \). The next time that the prime 739601 divides \( 4k^2 + 1 \) is for \( k = 739601 - 430 = 739171 \). Hence \( P_n \) is not a square for \( 430 \leq n \leq 739170 \).

Therefore \( \prod_{k=1}^{n} (4k^2 + 1) \) is not a perfect square. The proof that \( \prod_{k=1}^{n} (2k(k - 1) + 1) \) is not a perfect square is completely similar. This completes the proof of Theorem 2. \( \square \)

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**References**


[3] P. Dusart, The \( k \)th prime is greater than \( k(\ln k + \ln \ln k - 1) \) for \( k \geq 2 \), Math.Comput. 68 (1999), 411-415.