SOME ARITHMETIC PROPERTIES OF OVERPARTITION K-TUPLES

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Abstract

Recently, Lovejoy introduced the construct of overpartition pairs which are a natural generalization of overpartitions. Here we generalize that idea to overpartition k-tuples and prove several congruences related to them. We denote the number of overpartition k-tuples of a positive integer n by \( \overline{p}_k(n) \) and prove, for example, that for all \( n \geq 0 \), \( \overline{p}_{n-1}(tn + r) \equiv 0 \) (mod \( t \)) where \( t \) is prime and \( r \) is a quadratic nonresidue mod \( t \).

1. Introduction

As defined by Corteel and Lovejoy [5], an overpartition of a positive integer \( n \) is a non-increasing sequence of natural numbers whose sum is \( n \) in which the first occurrence of a part may be overlined. For example, the overpartitions of the integer 3 are

\[ 3, \overline{3}, 2 + 1, \overline{2} + \overline{1}, \overline{2} + 1, 2 + \overline{1}, 1 + 1 + 1, \overline{1} + \overline{1} + 1. \]

The number of overpartitions of a positive integer \( n \) is denoted by \( \overline{p}(n) \), with \( \overline{p}(0) = 1 \) by definition. Thus \( \overline{p}(3) = 8 \) from the above example. As noted in Corteel and Lovejoy [5], the generating function for overpartitions is

\[
\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n}.
\]

As the topic of overpartitions has already been examined rather thoroughly [3, 4, 5, 6, 7, 8, 10, 11], we look to new constructions. One such construction is that of an overpartition pair of a positive integer \( n \), defined by Lovejoy [9] as a pair of overpartitions wherein the sum of all listed parts is \( n \). For example, the overpartition pairs of 2 are

\[
(2 : \emptyset), (\overline{2} : \emptyset), (\emptyset : \overline{2}), (\emptyset : \overline{\overline{2}}), (1 + 1 : \emptyset), (\overline{1} + 1 : \emptyset), (\emptyset : 1 + 1), (\emptyset : \overline{1 + 1}), (1 : \overline{1}), (\overline{1} : 1), (\overline{1} : \overline{1}), (\overline{1} : 1).
\]
Lovejoy denoted the number of overpartition pairs of a positive integer \( n \) by \( \mathcal{pp}(n) \), with \( \mathcal{pp}(0) = 1 \) by definition. Thus \( \mathcal{pp}(2) = 12 \) from the above example. Following lines similar to that for overpartitions, the generating function for overpartition pairs is

\[
\sum_{n=0}^{\infty} \mathcal{pp}(n)q^n = \prod_{n=1}^{\infty} \left( \frac{1 + q^n}{1 - q^n} \right)^2.
\]

Several arithmetic properties of both overpartitions and their pairs have appeared in the literature. Since our interest here is primarily on congruence properties, there are a few theorems that are especially noteworthy. The first one is straightforward and proven intuitively.

**Theorem 1.** For all \( n > 0 \), \( \mathcal{p}(n) \equiv 0 \pmod{2} \).

Next we have a theorem easily proven using results of Mahlburg [10].

**Theorem 2.** For all \( n > 0 \),

\[
\mathcal{p}(n) \equiv \begin{cases} 
2 \pmod{4} & \text{if } n \text{ is a square}, \\
0 \pmod{4} & \text{otherwise}.
\end{cases}
\]

Several other congruences in arithmetic progressions were proven by Hirschhorn and Sellers. For example, the following were proven in [7].

**Theorem 3.** For all \( n \geq 0 \),

\[
\mathcal{p}(5n + 2) \equiv 0 \pmod{4}, \\
\mathcal{p}(5n + 3) \equiv 0 \pmod{4}, \\
\mathcal{p}(4n + 3) \equiv 0 \pmod{8}, \\
\text{and } \mathcal{p}(8n + 7) \equiv 0 \pmod{64}.
\]

Also, Hirschhorn and Sellers [6] proved that \( \mathcal{p}(n) \) satisfies congruences modulo non-powers of 2 by proving the following:

**Theorem 4.** For all \( n \geq 0 \) and all \( \alpha \geq 0 \), \( \mathcal{p}(9^{\alpha}(27n + 18)) \equiv 0 \pmod{12} \).

Finally, we note a theorem proven by Bringmann and Lovejoy [2]. This result provides much inspiration for the main result in the next section.

**Theorem 5.** For all \( n \geq 0 \), \( \mathcal{pp}(3n + 2) \equiv 0 \pmod{3} \).

We now introduce a generalization of overpartition pairs. An overpartition \( k \)-tuple of a positive integer \( n \) is a \( k \)-tuple of overpartitions wherein all listed parts sum to \( n \). We denote the number of overpartition \( k \)-tuples of \( n \) by \( \mathcal{p}_{k}(n) \), with \( \mathcal{p}_{k}(0) = 1 \) by
The aim of this note is to prove several congruence properties for families of overpartition \( k \)-tuples. In the process, we will prove several natural generalizations of results quoted above.

2. Results for Overpartition \( k \)-Tuples

Our first theorem of this section provides a natural generalization of Bringmann and Lovejoy’s Theorem 5 above. Moreover, the proof technique is extremely elementary, making this a very satisfying result.

**Theorem 6.** For all \( n \geq 0 \), \( p_{l-1}(tn + r) \equiv 0 \pmod{t} \), where \( t \) is an odd prime and \( r \) is a quadratic nonresidue \( \pmod{t} \).

**Remarks.** First, note that the \( t = 3 \) case of this theorem is exactly Theorem 5. Secondly, note that, for each odd prime \( t \), this theorem provides \( \frac{t-1}{2} \) congruence properties for \( p_{l-1}(n) \).

**Proof.** Consider the following generating function manipulations:

\[
\sum_{n=0}^{\infty} p_{l-1}(n)q^n = \prod_{i=1}^{\infty} \left( \frac{1 + q^i}{1 - q^i} \right)^{t-1} = \left[ \prod_{i=1}^{\infty} \frac{1 + q^i}{1 - q^i} \right]^{t-1} = \left[ \prod_{i=1}^{\infty} \frac{1 + q^{ti}}{1 - q^{ti}} \right] \left[ \prod_{i=1}^{\infty} \frac{1 - q^{ti}}{1 + q^{ti}} \right] \equiv \sum_{m=0}^{\infty} p(m)q^{tm} \prod_{i=1}^{\infty} \frac{1 - q^{ti}}{1 + q^{ti}} = \sum_{m=0}^{\infty} p(m)q^{tm} \sum_{s=-\infty}^{\infty} (-1)^s q^{2s^2} \text{ thanks to Gauss [1, Cor. 2.10].}
\]
But note that $tn + r$ can never be represented as $tm + s^2$ for some integers $m$ and $s$ if $r$ is a quadratic nonresidue mod $t$. This implies that $\overline{p}_{n-1}(tn + r) \equiv 0 \pmod{t}$ for all $n \geq 0$.

The next theorem is a broad generalization of Theorem 1. It is found with proof in [12], but is included here for the sake of completeness. We require a brief technical lemma.

**Lemma 7.** Let $m$ be a nonnegative integer. For all $1 \leq n \leq 2^m$,

$$\binom{2^m}{n}^2 \equiv 0 \pmod{2^{m+1}}.$$  

**Proof.** Let $\text{ord}_2(N)$ be the exponent of the highest power of 2 dividing $N$. Thus, for example, $\text{ord}_2(8) = 3$ while $\text{ord}_2(80) = 4$. To prove Lemma 7, we need to prove that

$$\text{ord}_2 \left( \binom{2^m}{n}^2 \right) \geq m + 1. \quad (1)$$

Note that

$$\text{ord}_2 \left( \binom{2^m}{n}^2 \right) = \text{ord}_2 \left( \frac{2^m(2^m-1)(2^m-2)\cdots(2^m-(n-1))}{n!} \cdot 2^n \right) \geq \text{ord}_2 \left( \frac{2^{m+n}}{n!} \right) = m + n - \text{ord}_2(n!) = m + n - \left( \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \cdots \right)$$

where $\lfloor x \rfloor$ is the floor function of $x$.

Now assume $n = c_02^0 + c_12^1 + \cdots + c_t2^t$ where each $c_i \in \{0, 1\}$. Then

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \cdots = c_12^0 + c_22^1 + \cdots + c_t2^{t-1} + c_22^0 + c_32^1 + \cdots + c_t2^{t-2} + c_32^0 + c_42^1 + \cdots c_t2^{t-3} + \cdots + c_t2^0$$

$$= (2 - 1)c_1 + (2^2 - 1)c_2 + (2^3 - 1)c_3 + \cdots + (2^t - 1)c_t = n - (c_0 + c_1 + c_2 + \cdots + c_t) \leq n - 1.$$
since at least one of the $c_i$ must equal 1. Therefore,

$$ord_2 \left( \binom{2m}{n}2^n \right) \geq m + n - \left( \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \frac{n}{8} + \cdots \right)$$
$$\geq m + n - (n - 1)$$
$$= m + 1.$$

This is the desired result as noted in (1) above.

We are now in a position to prove the following theorem:

**Theorem 8.** Let $k = (2^m)r$, where $m$ is a nonnegative integer and $r$ is odd. Then, for all positive integers $n$, we have $p_k(n) \equiv 0 \pmod{2^{m+1}}$.

**Proof.**

$$\sum_{n=0}^{\infty} \overline{p}_k(n)q^n = \prod_{i=1}^{\infty} \left[ \frac{1 + q^i}{1 - q^i} \right]^k$$
$$= \prod_{i=1}^{\infty} \left[ \frac{1 + q^i}{1 - q^i} \right]^{(2^m)r}$$
$$= \left( \prod_{i=1}^{\infty} \left[ \frac{1 + q^i}{1 - q^i} \right]^{2^m} \right)^r$$
$$= \left( \prod_{i=1}^{\infty} \left[ \frac{1 + \sum_{n=1}^{2^m} \binom{2m}{n}2^n \left( \frac{q^i}{1 - q^i} \right)^n \right] \right)^r$$
$$\equiv 1 \pmod{2^{m+1}} \text{ by Lemma 7.}$$

The following theorem is inspired by Theorem 2. As with Theorem 8, it primarily hinges upon the use of the binomial theorem.

**Theorem 9.** Let $k = (2^m)r$, $m > 0$ and $r$ is odd. Then, for all $n \geq 1$,

$$\overline{p}_k(n) \equiv \begin{cases} 
2^{m+1} \pmod{2^{m+2}} & \text{if } n \text{ is a square or twice a square,} \\
0 \pmod{2^{m+2}} & \text{otherwise.}
\end{cases}$$
Proof. We prove this result by induction on $m$.

**Basis Step.** Let $m = 1$. We must show that

$$p_{2r}(n) \equiv \begin{cases} 4 \pmod{8} & \text{if } n \text{ is a square or twice a square}, \\ 0 \pmod{8} & \text{otherwise}. \end{cases}$$

$$\sum_{n=0}^{\infty} p_{2r}(n)q^n = \prod_{i=1}^{\infty} \left( \frac{1 + q^i}{1 - q^i} \right)^{2r}$$

$$= \left( \sum_{n=0}^{\infty} p(n)q^n \right)^r$$

$$= \left( 1 + \sum_{\text{n > 0, square}} p(n)q^n + \sum_{\text{n > 0, not square}} p(n)q^n \right)^r$$

$$= 1 + 2 \left( \sum_{\text{n > 0, square}} p(n)q^n \right) + \left( \sum_{\text{n > 0, not square}} p(n)q^n \right)^2$$

$$+ 2 \left( \sum_{\text{n > 0, square}} p(n)q^n \right) \left( \sum_{\text{n > 0, not square}} p(n)q^n \right)^2$$

$$+ 2 \left( \sum_{\text{n > 0, not square}} p(n)q^n \right)^2 \equiv \left( \sum_{\text{n > 0, not square}} p(n)q^n \right)^2 \pmod{8}$$

From Theorem 2, we know that $p(n) \equiv 2$ or $6 \pmod{8}$ when $n$ is a square and $p(n) \equiv 0$ or $4 \pmod{8}$ otherwise. Since $2 \times 0, 2 \times 4, 6 \times 0, 6 \times 4, 0 \times 0, 0 \times 4$, and $4 \times 4$ are all congruent to $0 \pmod{8}$,

$$2 \left( \sum_{\text{n > 0, square}} p(n)q^n \right) \left( \sum_{\text{n > 0, not square}} p(n)q^n \right) \equiv 0 \pmod{8},$$

$$2 \left( \sum_{\text{n > 0, not square}} p(n)q^n \right) \equiv 0 \pmod{8},$$

and $$\left( \sum_{\text{n > 0, not square}} p(n)q^n \right)^2 \equiv 0 \pmod{8}.$$
This gives

\[ \sum_{n=0}^{\infty} p_{2r}(n)q^n = \left[ 1 + 2 \left( \sum_{n=1}^{\infty} p(n^2)q^n \right) + \left( \sum_{n=1}^{\infty} p(n^2)q^n \right)^2 \right]^r \pmod{8} \]

\[ \equiv \left[ 1 + 4 \left( \sum_{n=1}^{\infty} q^{n^2} \right) + 4 \left( \sum_{n=1}^{\infty} q^{n^2} \right)^2 \right]^r \pmod{8} \]

again thanks to Theorem 2.

Given that \((q^{n_1} + q^{n_2} + \cdots)^2 = (q^{2n_1} + q^{2n_2} + \cdots) + 2(q^{n_1+n_2} + \cdots)\), we then have

\[ \sum_{n=0}^{\infty} p_{2r}(n)q^n = \left[ 1 + 4 \left( \sum_{n=1}^{\infty} q^{n^2} \right) + 4 \left( \sum_{n=1}^{\infty} q^{n^2} + 2 \sum_{n_1, n_2 > 0, n_1 \neq n_2} q^{n_1^2+n_2^2} \right) \right]^r \pmod{8} \]

\[ \equiv \left[ 1 + 4 \left( \sum_{m=1}^{\infty} q^{m^2} + \sum_{n=1}^{\infty} q^{2n^2} \right) \right]^r \pmod{8} \]

\[ = \sum_{j=0}^{\infty} \binom{r}{j} 4^j \left( \sum_{m=1}^{\infty} q^{m^2} + \sum_{n=1}^{\infty} q^{2n^2} \right)^j \]

\[ \equiv 1 + 4 \left( \sum_{n=1}^{\infty} q^{n^2} + \sum_{n=1}^{\infty} q^{2n^2} \right) \pmod{8} \text{ since } r \text{ is odd.} \]

This proves the result needed for the basis step.

**Induction Step.** Assume that

\[ p_{(2^m)^r}(n) = \begin{cases} 2^{m+1} \pmod{2^{m+2}} & \text{if } n \text{ is a square or twice a square,} \\ 0 \pmod{2^{m+2}} & \text{otherwise.} \end{cases} \]

We must show that

\[ p_{(2^{m+1})^r}(n) = \begin{cases} 2^{m+2} \pmod{2^{m+3}} & \text{if } n \text{ is a square or twice a square,} \\ 0 \pmod{2^{m+3}} & \text{otherwise.} \end{cases} \]

Consider the generating function for \( p_{2^{m+1}}(n) \):
\[
\sum_{n=0}^{\infty} \overline{p}_{(2^m+1)r}(n)q^n = \prod_{i=1}^{\infty} \left( \frac{1+q^i}{1-q^i} \right)^{2^m+1}
\]
\[
= \left( \prod_{i=1}^{\infty} \left( \frac{1+q^i}{1-q^i} \right)^{2^m} \right)^2
\]
\[
= \left( \sum_{n=0}^{\infty} \overline{p}_{(2^m)r}(n)q^n \right)^2
\]
\[
= \left( 1 + \sum_{n>0} \overline{p}_{(2^m)r}(n)q^n + \sum_{n>0} \overline{p}_{(2^m)r}(n)q^n \right)^2
\]
\[
= 1 + 2 \left( \sum_{n>0} \overline{p}_{(2^m)r}(n)q^n \right) + \left( \sum_{n>0} \overline{p}_{(2^m)r}(n)q^n \right)^2
\]
\[
+ 2 \left( \sum_{n>0} \overline{p}_{(2^m)r}(n)q^n \right) \left( \sum_{n>0} \overline{p}_{(2^m)r}(n)q^n \right)
\]
\[
+ 2 \left( \sum_{n>0} \overline{p}_{(2^m)r}(n)q^n \right) + \left( \sum_{n>0} \overline{p}_{(2^m)r}(n)q^n \right)^2
\].

Using a very similar argument to that of the basis step, we use the induction hypothesis to conclude that

\[
\sum_{n=0}^{\infty} \overline{p}_{(2^m+1)r}(n)q^n \equiv 1 + 2 \sum_{n=1}^{\infty} \overline{p}_{(2^m)r}(n^2)q^{n^2} + \sum_{s=1}^{\infty} \overline{p}_{(2^m)r}(2s^2)q^{2s^2}
\]
\[
+ \left( \sum_{n=1}^{\infty} \overline{p}_{(2^m)r}(n^2)q^{n^2} + \sum_{s=1}^{\infty} \overline{p}_{(2^m)r}(2s^2)q^{2s^2} \right)^2 \pmod{2^m+3}
\]
\[
\equiv 1 + 2 \left( \sum_{n=1}^{\infty} \overline{p}_{(2^m)r}(n^2)q^{n^2} + \sum_{s=1}^{\infty} \overline{p}_{(2^m)r}(2s^2)q^{2s^2} \right) \pmod{2^m+3}.
\]
We know that all coefficients of the last term are congruent to $2^{m+1}$ or $2^{m+1} + 2^{m+2} \pmod{2^{m+3}}$ from the induction hypothesis. But the last term is multiplied by 2. So then all coefficients are congruent to $2^{m+2} \pmod{2^{m+3}}$ or $2^{m+2} + 2^{m+3} \equiv 2^{m+2} \pmod{2^{m+3}}$, which implies

$$\sum_{n=0}^{\infty} p_{(2^{m+1})}^{(2^{m+1})} q^n \equiv 1 + 2^{m+2} \left( \sum_{n=1}^{\infty} q^{n^2} + \sum_{s=1}^{\infty} q^{2s^2} \right) \pmod{2^{m+3}}.$$  

This completes the induction and proves the theorem. □

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