A NOTE ON THE EXACT EXPECTED LENGTH OF THE $k$TH PART OF A RANDOM PARTITION

Kimmo Eriksson
School of Education, Culture and Communication, Mälardalen University,
Västerås, SE-72123, Sweden
kimmo.eriksson@mdh.se

Received: 11/4/09, Revised: 1/29/10, Accepted: 3/3/10, Published: 6/11/10

Abstract
Kessler and Livingstone proved an asymptotic formula for the expected length of the largest part of a partition drawn uniformly at random. As a first step they gave an exact formula expressed as a weighted sum of Euler’s partition function. Here we give a short bijective proof of a generalization of this exact formula to the expected length of the $k$th part.

1. Results

By $\lambda \vdash n$ we will mean that $\lambda$ is a partition of $n$. This means that $\lambda$ is a finite non-increasing sequence of positive integers, $\lambda_1 \geq \cdots \geq \lambda_N > 0$, which sums to $n$. The number of partitions of $n$ is Euler’s famous partition function $p(n)$, with $p(0) = 1$ by convention.

Corteel et al. [1] mention a well-known partition identity attributed to Stanley: The expected number of different part sizes of a uniformly drawn partition $\lambda \vdash n$ is

$$\frac{1}{p(n)} \sum_{\ell \geq 1} \ell \cdot p_\delta(n, \ell) = \frac{1}{p(n)} \sum_{m=0}^{n-1} p(m).$$

(1)

Here, $p_\delta(n, \ell)$ denotes the number of partitions of $n$ with exactly $\ell$ different part sizes. The combinatorial proof in [1] is very simple: For any partition of $m = 0, 1, \ldots, n-1$, create a partition of $n$ by adjoining a part of size $n - m$. In so doing, any given partition of $n$ is created in as many copies as it has different part sizes.

First observe that this proof immediately generalizes to give a formula for the expected number of different part sizes $\geq k$ (that is, not counting any parts of size less than $k$):

$$\frac{1}{p(n)} \sum_{\ell \geq 1} \ell \cdot p_\delta(n, \ell, k) = \frac{1}{p(n)} \sum_{m=0}^{n-k} p(m),$$

(2)
where \( p_\ell(n, \ell, k) \) denotes the number of partitions of \( n \) with exactly \( \ell \) different part sizes \( \geq k \).

In this note we will make a similar generalization, with a combinatorial proof of the same flavor as above, of a formula of Kessler and Livingstone [3] for the expected length of the largest part \( \lambda_1 \) (or, equivalently, the number of parts) of a partition \( \lambda \vdash n \) drawn uniformly at random:

\[
E(\lambda_1) = \frac{1}{p(n)} \sum_{\lambda \vdash n} \lambda_1 = \frac{1}{p(n)} \sum_{m=1}^{n} p(n-m) \cdot \#\{d|m\},
\]

where \( \#\{d|m\} \) denotes the number of divisors of \( m \). Kessler and Livingstone used generating functions to prove (3). They then used this formula as a stepping stone toward an asymptotic formula for \( E(\lambda_1) \). For the large and interesting literature on asymptotic formulas for parts of integer partitions, we refer to Fristedt [2] and Pittel [4]. Here we focus on the finite formula (3). We shall present a simple combinatorial proof that immediately generalizes to the expected length of the \( k \)th longest part, \( \lambda_k \):

\[
E(\lambda_k) = \frac{1}{p(n)} \sum_{\lambda \vdash n} \lambda_k = \frac{1}{p(n)} \sum_{m=1}^{n} p(n-m) \cdot \#\{d|m : d \geq k\}.
\]

**Lemma 1** Let \( \lambda \) be any integer partition with \( k \)th part \( \lambda_k > 0 \). Then \( \lambda_k \) is also the number of pairs of integers \( r \geq 1 \) and \( d \geq k \) such that subtracting \( r \) from each of the \( d \) largest parts of \( \lambda \) results in a new partition.

**Proof.** Let \( N \) be the number of parts of \( \lambda \), and temporarily define \( \lambda_{N+1} = 0 \). After subtracting \( r \) from each of the \( d \) largest parts of \( \lambda \), what remains is a partition if and only if \( \lambda_d - r \geq \lambda_{d+1} \). Thus for each value of \( d \geq k \) we have \( \lambda_d - \lambda_{d+1} \) possible values of \( r \). The total number of possibilities is

\[
(\lambda_k - \lambda_{k+1}) + (\lambda_{k+1} - \lambda_{k+2}) + \cdots + (\lambda_N - \lambda_{N+1}),
\]

which simplifies to \( \lambda_k - \lambda_{N+1} = \lambda_k \). \( \square \)
Figure 1 illustrates the lemma.

Proof of (4). For any partition of $n - m$, with $m = 1, \ldots, n$, and any divisor $d \geq k$ of $m$, create a partition of $n$ by adding the integer $r = m/d \geq 1$ to each of the $d$ largest parts. In so doing, any given partition $\lambda$ of $n$ is created in exactly $\lambda k$ copies according to the lemma. □

Acknowledgments

This research was supported by the Swedish Research Council.

References


