ALTERNATIVE PROOFS ON THE 2-ADIC ORDER OF STIRLING NUMBERS OF THE SECOND KIND

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Abstract
An interesting 2-adic property of the Stirling numbers of the second kind $S(n, k)$ was conjectured by the author in 1994 and proved by De Wannemacker in 2005: $\nu_2(S(2^n, k)) = d_2(k) - 1, 1 \leq k \leq 2^n$. It was later generalized to $\nu_2(S(c2^n, k)) = d_2(k) - 1, 1 \leq k \leq 2^n, c \geq 1$ by the author in 2009. Here we provide full and two partial alternative proofs of the generalized version. The proofs are based on non-standard recurrence relations for $S(n, k)$ in the second parameter and congruential identities.

1 Introduction

The study of $p$-adic properties of Stirling numbers of the second kind offers many challenging problems. Let $k$ and $n$ be positive integers, and let $d_2(k)$ and $\nu_2(k)$ denote the number of ones in the binary representation of $k$ and the highest power of two dividing $k$, respectively. Lengyel [5] proved that

$$\nu_2(S(2^n, k)) = d_2(k) - 1$$

(1)

for all sufficiently large $n$ (e.g., $k - 2 \leq n$), and conjectured that $\nu_2(S(2^n, k)) = d_2(k) - 1$, for all $k : 1 \leq k \leq 2^n$ which was proved in

Theorem 1. ([3], Theorem 1) Let $k, n \in \mathbb{N}$ and $1 \leq k \leq 2^n$. Then we have

$$\nu_2(S(2^n, k)) = d_2(k) - 1.$$  

(2)

At the very heart of the proof, there is an appealing recurrence for the Stirling numbers of the second kind involving a double summation

$$S(n + m, k) = \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{j}{i} \frac{(k - i)!}{(k - j)!} S(n, k - i) S(m, j).$$

(3)
The generalization of Theorem 1 and De Wannemacker’s proof can be found in [7].

**Theorem 2.** ([7]) Let $c, k, n \in \mathbb{N}$ and $1 \leq k \leq 2^n$. Then

$$\nu_2(S(c2^n, k)) = d_2(k) - 1.$$  

In this paper we use Kummer’s theorem on the $p$-adic order of binomial coefficients.

**Theorem 3.** (Kummer (1852)) The power of a prime $p$ that divides the binomial coefficient $\binom{n}{k}$ is given by the number of carries when we add $k$ and $n-k$ in base $p$. In another form, $\nu_p\left(\binom{n}{k}\right) = \frac{n-d_p(n)}{p-1} - \frac{k-d_p(k)}{p-1} - \frac{n-k-d_p(n-k)}{p-1}$ with $d_p(n)$ being the sum of the digits of $n$ in its base $p$ representation. In particular, $\nu_2\left(\binom{n}{k}\right) = d_2(k) + d_2(n-k) - d_2(n)$ represents the carry count in the addition of $k$ and $n-k$ in base 2.

We will also need

**Theorem 4.** ([3], Theorem 3) Let $k, n \in \mathbb{N}$ and $1 \leq k \leq n$. Then

$$\nu_2(S(n, k)) \geq d_2(k) - d_2(n).$$  

This can be proven by an easy induction proof. Note that in general,

**Theorem 5.** ([6]) For every prime $p \geq 3$ and integer $k: 1 \leq k \leq n-1$,

$$\nu_p(S(n, k)) \geq \frac{d_p(k) - d_p(n) - (n-k)(p-2)}{p-1} + 1.$$  

The main goal of this paper is to suggest alternative methods for proving 2-adic properties of the Stirling numbers of the second kind. In Section 2 we discuss some partial proofs of Theorem 2 while full proofs of Theorems 1 and 2 are presented in Section 3. It is remarkable that both known proofs of Theorems 1 and 2 are based on recurrence relations on $S(n, k)$ in the second parameter such as (3) and (12) or its generalization (13).

## 2 Preliminaries and Partial Answers

In this section we provide alternative partial proofs of Theorem 2 for two sets of values of $k$ that are smaller than the full range $\{1, 2, \ldots, 2^n\}$. The proofs and how the tools, identity (6) and Theorem 8, are used seem to be new.
The two sets are defined by \( k \leq n \) and \( d_2(k) \leq \nu_2(k) \). Their respective cardinalities are \( n \) and the \((n+1)\)st Fibonacci number \( F_{n+1} \). In fact, by counting all values \( k \) with a fixed number \( s = d_2(k) \) of ones in their binary representations (so that \( s \leq \nu_2(k) \)), we find that there are \( \binom{n-s}{2} \) such \( ks \) if \( s \geq 2 \) and \( \binom{n}{2} \) powers of two otherwise. We get that

\[
| \{ k \mid 1 \leq k \leq 2^n \text{ and } d_2(k) \leq \nu_2(k) \} | = \binom{n}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \binom{n-4}{4} + \cdots = F_{n+1}, \text{ if } n \geq 1. 
\]

Let \( \pi(k; p^N) \) denote the minimum period of the sequence of Stirling numbers \( \{S(n,k)\}_{n\geq k} \text{ mod } p^N \). Kwong [4] proved the following.

**Theorem 6.** ([4]) For \( k > \max\{4, p\} \), \( \pi(k; p^N) = (p - 1)p^N + l_p(k) - 2 \), where \( p^{l_p(k) - 1} < k \leq p^{l_p(k)} \), i.e., \( l_p(k) = \left\lfloor \log_p k \right\rfloor \).

Based on the periodicity property and Euler’s theorem we can obtain:

**Theorem 7.** ([5], Theorem 2) Let \( c \) and \( n \) be non-negative integers, with \( c \) odd. If \( 1 \leq k \leq n + 2 \) then \( \nu_2(k!S(c2^n, k)) = k - 1 \), i.e., \( \nu_2(S(c2^n, k)) = d_2(k) - 1 \).

The latter theorem can be proven in a slightly weakened form by replacing \( k \leq n + 2 \) with \( k \leq n \) as is shown in the following proof.

**Proof.** We use the identity (cf. [8, identity (188) on p. 496])

\[
\sum_{d|N} \mu(d)k!S\left(\frac{N}{d}, k\right) \equiv 0 \text{ mod } N, \quad (6)
\]

for any positive integers \( k \) and \( N \), and \( \mu \) denoting the Moebius \( \mu \)-function. Indeed, we set \( N = 2^n, n \geq k \), and get that

\[
k!S(2^n, k) - k!S(2^{n-1}, k) \equiv 0 \text{ mod } 2^n. \quad (7)
\]

As above, by periodicity and Euler’s theorem, we know that \( \nu_2(k!S(2^n, k)) = k - 1 \) for any sufficiently large \( n \), and thus, by (7), we immediately have that it holds for any \( n \geq k \). This argument easily generalizes to \( S(c2^n, k) \) with any \( c \geq 1 \) odd; however, there will be \( 2^{\omega(c) + 1} \) terms of the form \( \pm k!S(c'2^n, k) \) or \( \pm k!S(c'2^{n-1}, k) \) in (7) where \( c' \geq 1 \) is a divisor of \( c \) and \( \omega(c) \) denotes the number of different prime factors of \( c \). The proof can be completed by an induction on \( \omega(c) \). \( \square \)
Another special case can be treated by the following theorem proved by Chan and Manna [2] in a recent paper.

**Theorem 8.** ([2], Theorem 4.2) Let $a, m$, and $n$ be positive integers with $m \geq 3$ and $n \geq a^{2m} + 1$. Then

$$S(n, a^{2m}) = a^{2m-1} \left( \left\lfloor \frac{n-1}{2} \right\rfloor - a^{2m-2} - 1 \right)$$

$$+ \frac{1 + (-1)^n}{2} \left( \frac{n}{2} - a^{2m-2} - 1 \right) \mod 2^m. \quad (8)$$

This guarantees that we can determine $\nu_2(S(2^n, k))$ for any $k$ with at least as many zeros at the end of its binary representation as the number of ones in it.

**Theorem 9.** Let $k, n \in \mathbb{N}$ and $1 \leq k \leq 2^n$ with $\max\{3, d_2(k)\} \leq \nu_2(k)$. Then $\nu_2(S(2^n, k)) = d_2(k) - 1$.

**Proof.** We replace $n$ by $2^n$ in Theorem 8 and write $k$ as $k = a^{2m}$ with some integer $a > 0$. We assume that $m \geq 3$ and $m \geq d_2(a)$, and $k = a^{2m} \leq 2^n$, i.e., $n \geq n_0 = \lceil \log_2(a^{2m}) \rceil$. Without loss of generality, we can assume that $a$ is odd and $m = \nu_2(k)$; otherwise, we rewrite $a^{2m}$ as $a'^{2m'}$ with $a'$ odd and $m' > m \geq d_2(a)$. Both (9) and (10) hold with $a'$ and $m'$ while $n$ and $n_0$ are kept unchanged.

Now we prove that

$$S(2^n, a^{2m}) = \left( \frac{2^{n-1} - a^{2m-2} - 1}{2^{n-1} - a^{2m-1}} \right) \mod 2^m \quad (9)$$

and

$$\nu_2(S(2^n, a^{2m})) = d_2(a) - 1 \quad (10)$$

by applying Theorem 8. Note that $\left\lfloor \frac{2^{n-1}}{2} \right\rfloor - a^{2m-2} - 1$ is even while $\left\lfloor \frac{2^{n-1}}{2} \right\rfloor - 2^{2m-1}$ is odd; thus, there is guaranteed at least one carry in the application of Theorem 3 to the binomial coefficient of the first term in (8). This proves (9) which can be further evaluated by the last part of Theorem 3. In fact, we get that

$$\nu_2(S(2^n, a^{2m})) = d_2(2^{n-1} - a^{2m-1}) + d_2(a^{2m-2} - 1) - d_2(2^{n-1} - a^{2m-2} - 1)$$

$$= (n - n_0 + (l_2(a) - d_2(a) - \nu_2(a) + 1))$$

$$+ (d_2(a) + \nu_2(a) - 1 + m - 2)$$

$$- (n - n_0 - 1 + (m - 2) + 1 + (l_2(a) - d_2(a) + 1))$$

$$= d_2(a) - 1 < m \quad (11)$$

with $l_2(a) = \lceil \log_2(a) \rceil$. \hfill \Box
Note that the above proof does not require any induction (although the proof of Theorem 8 uses induction). In addition, we can generalize the proof to obtain

**Theorem 10.** Let \( c, k, n \in \mathbb{N} \) and \( 1 \leq k \leq 2^n \) with \( \max\{3, d_2(k)\} \leq \nu_2(k) \). Then \( \nu_2(S(c2^n, k)) = d_2(k) - 1 \).

**Proof.** In fact, \( k = a2^m \leq 2^n \) implies that the nonzero binary digits of \( c2^n \) and \( a2^m \) avoid each other (perhaps with the exception of the rightmost one in \( c2^n \) when \( a = 1 \) and \( c \) is odd) and thus, (11) can be easily revised:

\[
\nu_2(S(c2^n, a2^m)) = d_2(c2^{n-1} - a2^{m-1}) + d_2(a2^{m-2} - 1) - d_2(c2^{n-1} - a2^{m-2} - 1)
\]

\[
= (n - n_o + (l_2(a) - d_2(a) - \nu_2(a) + 1) + d_2(c) + \nu_2(c) - 1)
\]

\[
+ (d_2(a) + \nu_2(a) - 1 + m - 2)
\]

\[
- (n - n_o - 1 + (m - 2) + 1 + (l_2(a) - d_2(a) + 1)
\]

\[
+ d_2(c) + \nu_2(c) - 1
\]

\[
= d_2(a) - 1 < m
\]

\[
\square
\]

3 **Main Result: Alternative Proofs of Theorems 1 and 2**

We now turn to another approach due to Agoh and Dilcher [1]. They developed an alternative recurrence relation for \( S(n + m, k) \) which relates this quantity to terms involving \( S(n, k')S(m, k - k') \) by means of a single summation rather than a double summation as in (3).

**Theorem 11.** ([1]) For \( r \geq \max\{k_1, k_2\} + 2 \), we have that

\[
\frac{k_1!k_2!(r - 1)!}{(k_1 + k_2 + 1)!} S(k_1 + k_2 + 2, r)
\]

\[
= \sum_{i=1}^{r-1} (i - 1)!(r - i - 1)! S(k_1 + 1, i) S(k_2 + 1, r - i).
\]

(12)

The paper [1] also contains a generalization of this theorem to \( s \geq 2 \) factors involving Stirling numbers on the right-hand side in a summation with \( s - 1 \) summation indices. Theorem 11 is a special case with \( s = 2 \).
We will use the generalization of (12) to \( r \geq 1 \), cf. [1, identity (6)]. It includes a correction term involving Bernoulli numbers

\[
\frac{(k - 1)!(m - 1)!(r - 1)!}{(k + m - 1)!} S(k + m, r)
\]

\[
= \sum_{i=1}^{r-1} (i - 1)!(r - i - 1)! S(k, i) S(m, r - i)
\]

\[
+ (r - 1)! \sum_{j=r}^{k+m-1} \left( (-1)^m \binom{k-1}{j-1} + (-1)^k \binom{m-1}{j-1} \right) \frac{B_{k+m-j}}{k + m - j} S(j, r)
\]

with \( B_n \) being the \( n \)th Bernoulli number.

Now we present an alternative proof of Theorem 1.

**Proof of Theorem 1.** We prove by induction on \( n \). The base case with \( n = 0 \) is trivial. We consider the equivalent form \( \nu_2(k!S(2^n, k)) = k - 1 \) of identity (1). Let us assume that \( \nu_2(k!S(2^t, k)) = k - 1 \) for any integers \( t \) and \( k \) such that \( 1 \leq t \leq n \) and \( 1 \leq k \leq 2^t \). We prove the statement for \( t = n + 1 \). We write \( k \) in its binary representation \( k = 2^{b_1} + 2^{b_2} + \cdots + 2^{b_{d_2(k)}} \) with \( 0 \leq b_1 < b_2 < \cdots < b_{d_2(k)} \). We have two cases according whether \( k \geq 2^n + 1 \) or not.

**Case 1.** First let us assume that

\[
2^n < k \leq 2^{n+1}.
\]

The assumption yields that \( b_{d_2(k)} = n \) except for \( k = 2^{n+1} \).

We use Theorem 11 with \( k_1 = k_2 = 2^n - 1, r \geq 2^n + 1 \), and switching from the notation \( r \) to \( k \). After slightly rewriting (12), we obtain

\[
(k - 1)!S(2^{n+1}, k) = \frac{(2^{n+1} - 1)!}{(2^n - 1)!^2} \sum_{i=1}^{k-1} \frac{1}{i(k - i)!} S(2^n, i)(k - i)!S(2^n, k - i).
\]

With \( N = 2^{n+1} \), the first factor on the right-hand side of (15) is

\[
\frac{(N - 1)!}{(N/2 - 1)!^2} = \left( \frac{N - 1}{N/2} \right) \frac{N}{2}
\]

and there is no carry in the addition of \( N/2 \) and \( N/2 - 1 \). This yields an overall 2-adic order of \( n \) for the whole expression.
We have two subcases. If \( k \) is odd then we note that \( i(k - i) \) in the denominator of (15) can decrease the 2-adic order, and the unique largest decrement results from setting \( i \) or \( k - i \) to \( 2^{b_{d_2}(k)} \). By the inductive hypothesis, the last four factors at the end of (15) contribute \((i - 1) + (k - i - 1) = k - 2\) to the 2-adic order. Hence, we get that

\[
\nu_2(k(k - 1)!S(2^{n+1}, k)) = \nu_2(k) + n - b_{d_2(k)} + 1 + (k - 2) \\
= n + k - 1 - b_{d_2(k)} = k - 1.
\]

(16)

If \( k \) is even and \( k \neq 2^{n+1} \) then the factor \( i(k - i) \) in the denominator of (15) decreases the 2-adic order the most if we set \( i \) or \( k - i \) to \( 2^{b_{d_2}(k)} \) which yields that the other factor is an odd multiple of \( 2^{\nu_2(k)} \). No other pair \((i, k - i)\) can reach this decrement. If \( i = k/2 \) then the corresponding term occurs only once, and the decrement is \( 2(\nu_2(k) - 1) \leq b_{d_2(k)} + \nu_2(k) - 2 \). Thus, the right-hand side of (16) changes, and we obtain

\[
\nu_2(k!S(2^{n+1}, k)) = \nu_2(k) + n - (b_{d_2(k)} + \nu_2(k)) + 1 + (k - 2) \\
= n + k - 1 - b_{d_2(k)} = k - 1.
\]

(17)

For \( k = 2^{n+1} \), since the factor \( i(k - i) \) decreases the 2-adic order the most if we set both \( i \) and \( k - i \) to \( 2^{b_{d_2}(k)} = 2^n \), we get

\[
\nu_2(k!S(2^{n+1}, k)) = \nu_2(k) + n - (b_{d_2(k)} - 1 + \nu_2(k) - 1) + (k - 2) \\
= n + k - b_{d_2(k)} = k - 1.
\]

Case 2. Now we assume that \( k \leq 2^n \) and have two subcases. First we discuss the case with \( k < 2^n \) provided that \( k \) is not a power of two then we consider the case in which \( k = 2^m, m \leq n \).

Since now \( k \leq 2^n \), we need the correction term in (13) which leads to the revised version of (15)

\[
k(k - 1)!S(2^{n+1}, k) = k(2^{n+1} - 1)! \sum_{i=1}^{k-1} \frac{1}{i(k - i)} S(2^n, i) (k - i)!S(2^n, k - i) \\
+ k(k - 1)! \frac{(2^{n+1} - 1)!}{(2^n - 1)!} \sum_{j=k}^{2^n} 2\left(\frac{2^n - 1}{j - 1}\right) B_{2^n+1-j} S(j, k)
\]

(18)

by setting \( k \) and \( m \) to \( 2^n \) and switching from \( r \) to \( k \) in (13). We proceed similarly to (16) and (17), but this time the correction term in (18) will determine the exact
2-adic order. Clearly, the factor \( \binom{2^n - 1}{j - 1} \) in the correction term is odd for any \( j, k \leq j \leq 2^n \), by Theorem 3.

If \( k < 2^n \) then \( b_{d_2(k)} \leq n - 1 \). If \( k \) is not a power of two then the right-hand sides of (16) and (17) become \( n + k - 1 - b_{d_2(k)} \geq k \). Therefore, the first term on the right-hand side of (18) contributes an integer multiple of \( 2^k \) to (18). On the other hand, the correction term of (18) will guarantee that \( \nu_2(k!S(2^{n+1}, k)) \) stays at \( k - 1 \). Indeed, the 2-adic order of the \( j \)th term of the correcting sum is at least \( (k - d_2(k)) + n + (1 + \nu_2(B_{2^{n+1-j}} - \nu_2(j)) + (d_2(k) - d_2(j)) \geq n + (k - 1) + (1 - \nu_2(j) - d_2(j)) = n + (k - 1) - d_2(j - 1) \) by Theorem 4 and the fact that \( \nu_2(B_n) \geq -1 \). For the smallest possible value we have that

\[
\min_{k \leq j \leq 2^n} n + (k - 1) - d_2(j - 1) = k - 1 \tag{19}
\]

taken uniquely at \( j = 2^n \). In this case the two inequalities above become equalities since \( \nu_2(S(2^n, k)) = d_2(k) - 1 \) and \( \nu_2(B_{2^n}) = -1 \). Thus, \( \nu_2(k!S(2^{n+1}, k)) = k - 1 \).

We are left with the subcases in which \( k \) is a power of two. The statement is trivially true for \( k = 1 \). If \( k = 2^m \) with \( 1 \leq m \leq n \) then \( b_{d_2(k)} = \nu_2(k) = m \) and the right-hand side of (17) changes to

\[
\nu_2(k) + n - (b_{d_2(k)} - 1 + \nu_2(k) - 1) + (k - 2) = n - m + k \geq k
\]

with \( \max_{1 \leq i \leq k - 1} \nu_2(i(k - i)) = b_{d_2(k)} - 1 + \nu_2(k) - 1 \) and the unique optimum is taken at \( i = k - i = 2^{m-1} \). For the correction term, (19) applies again with the same reasoning as above. \( \square \)

We can generalize the above proof to obtain an alternative proof of Theorem 2 although it requires a modified version of inequality (5) of Theorem 4, cf. \cite[Remark 2 and Theorem 6]{7} in a somewhat relaxed form:

**Theorem 12.** For \( c \geq 3 \) odd, we have

\[
\nu_2(S(c2^n, k)) \geq d_2(k) - 1, \ 1 \leq k \leq 2^{n+1}. \tag{20}
\]

Below, for any integer \( a \geq 1 \), we use the following simple fact that

\[
d_2(a - 1) = d_2(a) - 1 + \nu_2(a). \tag{21}
\]

This implies \( d_2(c2^n - 1) = d_2(c - 1) + n \) and thus,

\[
d_2(c2^{n+1} - 1) = d_2(c2^n - 1) + 1 = d_2(c) + \nu_2(c) + n. \tag{22}
\]
Proof of Theorem 2. We may assume that $c$ is an odd integer, otherwise we can factor $c$ into a power of two and an odd integer, and $k$ still satisfies $1 \leq k \leq 2^n$. We use induction on $c$ and $n$. Assume that $\nu_2(k!S(2^c,k)) = k-1, 1 \leq k \leq 2^c$, for all $1 \leq s \leq c$ and $0 \leq t \leq n$, and prove that it also holds for $t = n + 1$. Then we prove that it also holds for the odd number $s = c + 2$.

The base case with $c = 1$ is covered by the above proof of Theorem 1. Let us assume that $c \geq 3$. Clearly, $d_2(c) \geq 2$. The case with $n = 0$ is trivial since $\nu_2(S(c,1)) = 0$. Similarly to (18), we get

$$k(k-1)!S(2^{c+1},k)$$

$$= \frac{k(c^{2n+1} - 1)!}{(c^{2n} - 1)!^2} \sum_{i=1}^{c-1} \frac{1}{i!(k-1)!} \cdot i!S(c^{2^n},i)(k-i)!S(c^{2^n},k-i)$$

$$+ k(k-1)!\frac{(c^{2n+1} - 1)!}{(c^{2n} - 1)!^2} \sum_{j=k}^{c^n} 2^{j} \left( \frac{c^{2^n} - 1}{c^{2^n+1} - j} \right) B_{i,2^{n+1-j}} S(j,k)$$

(23)

by setting $k = m = c^{2^n}$ and switching from $r$ to $k$ in (13). We will see that the correction term in (23) determines the exact 2-adic order. In fact, the first term’s 2-adic order is at least

$$\nu_2(k) + (n-1 + d_2(c)) + k - 2$$

$$= \begin{cases} 
\lfloor \log_2 k \rfloor + \nu_2(k) - 1, & \text{if } c \geq 2 \text{ is odd or even but not a power of two} \\
2\nu_2(k) - 2, & \text{if } c \geq 2 \text{ is a power of two},
\end{cases}$$

by (22) and Theorem 12, thus it is at least $k$. Note that the first term disappears if $k = 1$, and the statement $\nu_2(S(c^{2n+1},1)) = 0$ is trivial.

If $j$ is odd then the corresponding Bernoulli number $B_{i,2^{n+1-j}}$ in the correction term (23) is 0. If $j$ is even then we define $A$ as the 2-adic order of the $j$th term, and we have that

$$A = \nu_2(k!) + \nu_2((c^{2n+1} - 1)! - 2\nu_2((c^{2n} - 1)!))$$

$$+ (1 + d_2(j-1) + d_2(c^{2n} - j) - d_2(c^{2n} - 1) - 1 - \nu_2(c^{2n+1} - j))$$

$$+ \nu_2(S(j,k))$$

$$= (k - d_2(k)) + c^{2n+1} - 1 - d_2(c^{2n+1} - 1) - 2(c^{2n} - 1 - d_2(c^{2n} - 1))$$

$$+ (d_2(j-1) + d_2(c^{2n} - j) - d_2(c^{2n} - 1) - \nu_2(c^{2n+1} - j))$$

$$+ \nu_2(S(j,k))$$
\begin{align*}
&= k + d_2(j - 1) + d_2(c2^n - j) - \nu_2(c2^{n+1} - j) + \nu_2(S(j, k)) - d_2(k) \\
&= k - 1 + \nu_2(j) + d_2(c2^n - j) - \nu_2(c2^{n+1} - j) + (\nu_2(S(j, k)) - d_2(k) + d_2(j))
\end{align*}
by \( \nu_2(B_{c2^{n+1} - j}) = -1, (21), \) and \( (22). \)

Now we prove that the last quantity is at least \( k - 1, \) and the unique value of \( j \) that achieves this lower bound is \( j = c \mod 2^\lfloor \log_2 c \rfloor, \) i.e., when we remove the most significant binary digit of \( c. \) We set \( j = c'2^{n+q} \) with \( c' \) odd and \( k \leq j \leq c2^n \) and identify four cases according to the value of \( q. \)

If \( -n \leq q < 0 \) then
\[
A \geq k - 1 + n + q + d_2(c2^{-q} - c') - (n + q) \geq k
\]
by \( (5) \) and since \( c' \neq c2^{-q}, \) i.e., \( j \neq c2^n. \) If \( q = 0, \) i.e., \( j = c'2^n, \) then
\[
A \geq k - 1 + n + d_2(c) - n + (d_2(k) - 1 - d_2(c) + d_2(c'))
\]
by Theorem 12. If \( q = 1 \) then \( 2c' < c \) and
\[
A = k - 1 + n + 1 + d_2(c - 2c') - \nu_2(c - c') - (n + 1) + (-1 + d_2(c'))
\]
by the induction hypothesis as \( c' < c \) and \( 1 \leq k \leq 2^{n+1} \) imply that \( \nu_2(S(c'2^{n+1}, k)) = d_2(k) - 1. \) It is easy to prove, e.g., by induction on the number of blocks of zeros in the binary representation of \( c, \) that \( A \) can reach the lower bound \( k - 1 \) exactly if \( c' \) is derived from \( c \) by removing its most significant binary digit. By the way, if \( c'' = c2^{\lfloor \log_2 c \rfloor - 1} \) with \( 0 \leq i \leq \lfloor \log_2 c \rfloor - 1, \) then \( d_2(c) - 1 + \nu_2(\binom{c}{2^i}) - \nu_2(c - c'') \) is equal to the number of ones in \( c2^{\lfloor \log_2 c \rfloor} - c''. \)

If \( q \geq 2 \) then \( (5) \) we get that
\[
A \geq k - 1 + n + q + d_2(c - c'2^q) - (n + 1) \geq k - 1 + q - 1 \geq k.
\]
The proof of \( \nu_2(k!S(c2^{n+1}, k)) = k - 1 \) for \( 1 \leq k \leq 2^{n+1} \) and \( n \geq 0 \) is complete for \( c, \) and now we can proceed with the next odd \( c. \)

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References


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