FINDING ALMOST SQUARES V

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Abstract
An almost square of type 2 is an integer \( n \) that can be factored in two different ways as \( n = a_1b_1 = a_2b_2 \) with \( a_1, a_2, b_1, b_2 \approx \sqrt{n} \). In this paper, we continue the study of almost squares of type 2 in short intervals and improve the \( 1/2 \) upper bound. We also draw connections with almost squares of type 1.

1. Introduction and Main Results
An almost square (of type 1) is an integer \( n \) that can be factored as \( n = ab \) with \( a, b \) close to \( \sqrt{n} \). For example, \( 9999 = 99 \times 101 \). We call an integer \( n \) an almost square of type 2 if it has two different such representations, \( n = a_1b_1 = a_2b_2 \) where \( a_1, b_1, a_2, b_2 \) are close to \( \sqrt{n} \). For example \( 99990000 = 9999 \times 10000 = 9900 \times 10100 \). Of course, this depends on what we mean by close. More precisely, for \( 0 \leq \theta \leq 1/2 \) and \( C > 0 \),

Definition 1 An integer \( n \) is a \((\theta, C)\)-almost square of type 1 if \( n = ab \) for some integers \( a < b \) in the interval \([n^{1/2} - Cn^\theta, n^{1/2} + Cn^\theta]\).

Definition 2 An integer \( n \) is a \((\theta, C)\)-almost square of type 2 if \( n = a_1b_1 = a_2b_2 \) for some integers \( a_1 < a_2 \leq b_2 < b_1 \) in the interval \([n^{1/2} - Cn^\theta, n^{1/2} + Cn^\theta]\).

In a series of papers [1], [2], [3], [4], the author was interested in finding almost squares of either types in short intervals. In particular, given \( 0 \leq \theta \leq \frac{1}{2} \), we want to find “admissible” exponent \( \phi_i \geq 0 \) (as small as possible) such that, for some constants \( C_{\theta,i}, D_{\theta,i} > 0 \), the interval \([x - D_{\theta,i}x^{\phi_i}, x + D_{\theta,i}x^{\phi_i}]\) contains a \((\theta, C_{\theta,i})\)-almost square of type \( i \) \((i = 1, 2)\) for all sufficiently large \( x \). These lead to the following

Definition 3 We let \( f(\theta) := \inf \phi_1 \) and \( g(\theta) := \inf \phi_2 \), where the infima are taken over all the “admissible” \( \phi_i \) \((i = 1, 2)\) respectively.
Clearly \( f \) and \( g \) are non-increasing functions of \( \theta \). It was conjectured (and partially verified) that

**Conjecture 4** For \( 0 \leq \theta \leq \frac{1}{2} \),

\[
f(\theta) = \begin{cases} 
\frac{1}{2}, & \text{if } 0 \leq \theta < \frac{1}{4}, \\
\frac{1}{2} - \theta, & \text{if } \frac{1}{4} \leq \theta \leq \frac{1}{2};
\end{cases}
\]

and

\[
g(\theta) = \begin{cases} 
does not exist, & \text{if } 0 \leq \theta < \frac{1}{4}, \\
1 - 2\theta, & \text{if } \frac{1}{4} \leq \theta \leq \frac{1}{2}.
\end{cases}
\]

In [3], it was proved that

**Theorem 5** For \( \frac{1}{4} \leq \theta \leq \frac{1}{2} \),

\[
g(\theta) \leq \begin{cases} 
\frac{5}{8}, & \text{if } \frac{1}{4} \leq \theta \leq \frac{5}{16}, \\
\frac{17}{32}, & \text{if } \frac{5}{16} \leq \theta \leq \frac{743}{2306}, \\
\frac{1}{2}, & \text{if } \frac{743}{2306} < \theta \leq \frac{1}{2}.
\end{cases}
\]

The purpose of this paper is to improve the \( \frac{1}{2} \) upper bound for \( g(\theta) \) in certain range of \( \theta \), namely

**Theorem 6** For \( \frac{1}{4} \leq \theta \leq \frac{1}{2} \), we have \( g(\theta) \leq 1 - \frac{3\theta}{7} \).

Combining the above two theorems, we have the following picture:
The thin downward sloping line is the conjectural lower bound $1 - 2\theta$ while the thick line segments above are the upper bounds from Theorems 5 and 6.

Furthermore, there are some connections between almost squares of type 1 and almost squares of type 2.

**Theorem 7** If Conjecture 4 is true for $f(\theta)$ when $\frac{1}{4} \leq \theta \leq \frac{1}{2}$, then

$$g(\theta) \leq \frac{3}{2} - 3\theta$$

for $\frac{1}{3} \leq \theta \leq \frac{1}{2}$.

**Notation.** Both $f(x) = O(g(x))$ and $f(x) \ll g(x)$ mean that $|f(x)| \leq Cg(x)$ for some constant $C > 0$.

2. Unconditional Result: Theorem 6

**Proof.** We shall use the fact: for any real number $x \geq 1$, there exists a perfect square $a^2$ such that $a^2 \leq x < (a + 1)^2$. Hence $|x - a^2| \ll \sqrt{x}$.

Given $x \geq 1$ sufficiently large. The almost square of type 2 close to $x$ we have in mind has the form

$$n = (G^2 - 1)(H^2 - h^2) = a_1b_1 = a_2b_2,$$

where $\{a_1, b_1\} = \{(G - 1)(H - h), (G + 1)(H + h)\}$ and $\{a_2, b_2\} = \{(G - 1)(H + h), (G + 1)(H - h)\}$.

Let $0 < \lambda < \frac{1}{4}$. We choose $G = [x^{1/4-\lambda}]$.

First, we approximate $\frac{x}{G^2 - 1}$ by $H^2$ where $H = \lfloor \frac{x}{G^2 - 1} \rfloor + 1$. Then $0 < H^2 - \frac{x}{G^2 - 1} \ll H$. One can check that

$$GH = G\left[\sqrt{\frac{x}{G^2 - 1}} + 1\right] = x^{1/2}\left(1 + O\left(\frac{1}{G^2}\right)\right) + O(G) = x^{1/2} + O(x^{2\lambda}) + O(x^{1/4-\lambda}).$$

Next, we approximate $H^2 - \frac{x}{G^2 - 1}$ by $h^2$ for some $0 < h \ll H^{1/2} \ll x^{1/8+\lambda/2}$. We can get within a distance $H^2 - \frac{x}{G^2 - 1} - h^2 \ll H^{1/2} \ll x^{1/8+\lambda/2}$. Therefore

$$|x - (G^2 - 1)(H^2 - h^2)| \leq \left|\frac{x}{G^2 - 1} - (H^2 - h^2)\right|G^2 \ll G^2x^{1/8+\lambda/2} \ll x^{5/8-3\lambda/2}. $$
The number \( n = (G^2 - 1)(H^2 - h^2) = (G - 1)(G + 1)(H - h)(H + h) \). Notice that
\[
(G - 1)(H - h) = GH - H - Gh + h
\]
\[
= x^{1/2} + O(x^{2\lambda}) + O(x^{1/4-\lambda})
\]
\[
+ O(x^{1/4+\lambda}) + O(x^{3/8-\lambda/2}) + O(x^{1/8+\lambda/2})
\]
\[
= x^{1/2} + O(x^{1/4+\lambda}) + O(x^{3/8-\lambda/2}).
\]

One can check that the same asymptotic holds for \((G - 1)(H + h), (G + 1)(H - h)\)
and \((G + 1)(H + h)\). When \( \lambda \geq \frac{1}{12} \), the first error term dominates. Thus, we just have found a \((\frac{1}{4} + \lambda, C)\)-almost square of type 2 within a distance \(O(x^{5/8-3\lambda/2})\) from \(x\) for some \(C > 0\).

Set \( \theta = \frac{1}{4} + \lambda \). The condition \( \frac{1}{12} \leq \lambda < \frac{1}{4} \) becomes \( \frac{1}{2} \leq \theta < \frac{3}{4} \). Meanwhile \( \frac{5}{8} - \frac{11\lambda}{2} = 1 - \frac{3\theta}{2} \). Therefore, for any \( \frac{1}{4} \leq \theta < \frac{1}{2} \), there exists a \((\theta, C)\)-almost square of type 2 within a distance \(O(x^{1-3\theta/2})\) from \(x\). So \( g(\theta) \leq 1 - \frac{2\theta}{3} \) when \( \frac{1}{4} \leq \theta < \frac{1}{2} \).

When \( \lambda = \frac{1}{4} \) (i.e. \( \theta = \frac{1}{2} \)), one simply uses \( G = 2 \) and the above argument works in the same way to give \( g(\frac{1}{2}) \leq \frac{1}{4} \).

\[\square\]

3. Connection to Almost Squares of Type 1: Theorem 7

Proof. In the previous section, we used an elementary method to approximate \( \frac{x}{G^2 - 1} \)
by \((H - h)(H + h)\), a \((\frac{1}{4}, C)\)-almost square of type 1, since \( h \ll H^{1/2} \ll \sqrt{\frac{x}{G^2 - 1}} \).
So one should expect to do better using \((\phi, C)\)-almost square of type 1 for some \( \frac{1}{4} \leq \phi \leq \frac{1}{2} \).

Again we choose \( G = \lfloor x^{1/4-\lambda} \rfloor \) and let \( H = \sqrt{\frac{x}{G^2 - 1}} \). By Conjecture 4 on \( f(\theta) \),
we can find a \((\phi, C)\)-almost square of type 1, say \( ab \), such that \( H - CH^{2\phi} \leq a \leq \phi \leq H + CH^{2\phi} \) and
\[
\left| \frac{x}{G^2 - 1} - ab \right| \ll \left( \frac{x}{G^2 - 1} \right)^{1/2-\phi+\epsilon} \ll x^{1/4-\phi/2+\lambda-2\lambda\phi+\epsilon}
\]
for \( x \) sufficiently large. Hence
\[
|x - (G - 1)(G + 1)ab| \ll x^{3/4-\lambda-\phi/2-2\lambda\phi+\epsilon}.
\]

(1)

Similar to the previous section, one has
\[
(G - 1)a = (G - 1)(H + O(H^{2\phi})) = GH - H + O(GH^{2\phi})
\]
\[
= x^{1/2} + O(x^{1/4+\lambda}) + O(x^{1/4-\lambda+\phi/2+2\lambda\phi}).
\]
The same is true for \((G - 1)b, (G + 1)a\) and \((G + 1)b\). One can check that \(\frac{1}{4} + \lambda \geq \frac{1}{4} - \lambda + \frac{\phi}{2} + 2\lambda\phi\) if and only if \(\lambda \geq \frac{2}{4 - 4\phi}\).

Let \(\theta = \frac{1}{4} + \lambda\) and \(\lambda = \frac{\phi}{4 - 4\phi}\). Then the exponent in (1) satisfies \(\frac{3}{2} - \lambda - \frac{\phi}{2} - 2\lambda\phi = 1 - \theta(1 + 2\phi)\). Therefore, for any \(\frac{1}{4} + \frac{\phi}{4 - 4\phi} \leq \theta \leq \frac{1}{2}\), there exists a \((\theta, C')\)-almost square of type 2 within a distance of \(O(x^{1 - \theta(1 + 2\phi) + \epsilon})\) from \(x\) for some \(C' > 0\).

Given \(\frac{1}{3} \leq \theta \leq \frac{1}{2}\), the bigger the \(\phi\), the better the above result. Since \(\frac{\phi}{4 - 4\phi}\) is an increasing function of \(\phi\), the biggest \(\phi\) we can use is when \(\frac{1}{3} + \frac{\phi}{4 - 4\phi} = \theta\). This gives \(\phi = 1 - \frac{1}{3\theta} \leq \frac{1}{2}\) as \(\theta \leq \frac{1}{2}\). Using this value of \(\phi\), we have a \((\theta, C')\)-almost square of type 2 within a distance of \(O(x^{3/2 - 3\theta + \epsilon})\) from \(x\). This proves Theorem 7 as \(\epsilon\) can be arbitrarily small.

**Remark.** The exponent \(\frac{3}{2} - 3\theta \to 0\) as \(\theta \to \frac{1}{2}\). However \(\frac{3}{2} - 3\theta\) is always greater than the conjectural value \(1 - 2\theta\) for \(g(\theta)\) which is no surprise as part of the almost square has the special form \(G^2 - 1\). It would be interesting to see how one could incorporate the extra degree of freedom, namely \(G^2 - g^2\) for some \(g\), for further improvements.

### References


