AN OBSERVATION ON THE EXTENSION OF ABEL’S LEMMA

Alexander E. Patkowski
Department of Mathematics, University of Regina, Regina, Saskatchewan, Canada
S4S 0A2
alexpatk@hotmail.com

Received: 2/27/10, Revised: 7/26/10, Accepted: 8/25/10, Published: 12/1/10

Abstract
We prove a general formula for a certain class of sums of tails. A new proof of some
known identities is given, and a general identity that appears to be the first of its
kind is established.

1. Introduction and Main Results
In the work of Andrews, Jiménez-Urroz and Ono [2], it was shown that sums over the
differences of q-products and truncated q-products have very interesting applications
to both partitions and generating functions for values of L-functions (see [5, 8] as
well). Series of this type are more casually termed sums of tails [5]. One elegant
example, given in Ramanujan’s “lost” notebook [4], is:

\[ \sum_{n=0}^{\infty} ((-q)_{\infty} - (-q)_n) = (-q)_{\infty} D(q) + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^{(n+1)/2}}{(-q)_n}, \] (1)

where

\[ D(q) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}, \]

and where the q-Pochhammer symbol [7] \((\xi)_n = (\xi; q)_n := \prod_{j=0}^{n-1} (1 - \xi q^j)\) has been
used. In [1], Andrews gave the first proof of (1) and a related identity found in
Ramanujan’s “lost” notebook [4]. Zagier [9] also studied series of this type after
Ramanujan, and his work gave some of the motivation for the results contained in
[2].

Proposition 2.1 of [1] is the key formula to obtaining these special sums of
tails identities, and its proof requires use of Abel’s lemma, which states that if
\(\lim_{n \to \infty} a_n = L\), then \(\lim_{t \to 1^-} (1 - t) \sum_{n=0}^{\infty} a_n t^n = L\). For a proof of Proposition
2.1, which we now state, see [4], Chapter 7, “Special Identities.”
(Proposition 2.1. [2]) Suppose that \( f(z) = \sum_{n=0}^{\infty} \alpha(n)z^n \) is analytic for \( |z| < 1 \). If \( \alpha \) is a complex number for which \[
\sum_{n=0}^{\infty} (\alpha - \alpha(n)) < +\infty
\]
and
\[
\lim_{n \to \infty} n(\alpha - \alpha(n)) = 0,
\]
then
\[
\lim_{z \to 1} \frac{d}{dz} (1 - z)f(z) = \sum_{n=0}^{\infty} (\alpha - \alpha(n)).
\]

Andrews and Freitas [3] have found how to generalize Proposition 2.1 of [2] to the \( p \)-th derivative. Two examples from [3], which are of interest to our study, are:
\[
\sum_{n=0}^{\infty} \left( \frac{1}{(q)_{\infty}} \frac{1}{{(q)_{n}}} \right)^2 = \frac{1}{(q)_{\infty}^2} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n+1/2} \frac{1 - (q)_n}{(q)_n} 1 - q^n, \tag{2}
\]
\[
\sum_{n=0}^{\infty} \left( \frac{(t)_n}{(q)_n} \frac{1}{{(q)_{n}}} \right)^2 = \frac{(t)_{\infty}^2}{(q)_{\infty}^2} \sum_{n=1}^{\infty} \frac{1}{(q)_n^2} \frac{(q/t)_n}{(q)_n} \left( \frac{(q)_n}{(t)_n} - 1 \right) \frac{1}{1 - q^n}. \tag{3}
\]

Here (2) is [3, pg.148, eq.(iii)] and a corollary of (3), and (3) is [3, pg.148, eq.(ii)]. As it turns out, these two identities are also a consequence of our main theorem, and can be viewed as a corollary of Proposition 2.1.

Theorem 1 Assuming the hypothesis of Proposition 2.1, we have for each integer \( r \geq 1 \),
\[
\sum_{n=0}^{\infty} (\alpha - \alpha(n))^r = -\sum_{k=1}^{r} \binom{r}{k} (\alpha)^{r-k} (-1)^k \sum_{n=0}^{\infty} (\alpha^k - \alpha^k(n)). \tag{4}
\]

Throughout this note we will be primarily concerned with applications to \( q \)-series, and so we will not run into any problems with our choices of \( \alpha(n) \).
2. Proof of Main Theorem

The proof of Theorem 1 essentially relies on Proposition 2.1, which is why one can view Theorem 1 as a corollary of Proposition 2.1. First, we need the binomial theorem [7, p. 25]:

\[(x+y)^r = \sum_{k=0}^{r} \binom{r}{k} x^{r-k} y^k. \quad (5)\]

Set \(\alpha = 0, \alpha(n) = \eta_n(r) := (\lambda - \lambda_n)^r\), and assume \(\eta_n(r)\) satisfies (i) and (ii), where \(\lambda_n\) also satisfies the hypothesis of Proposition 2.1 and \(\lim_{n \to \infty} \lambda_n = \lambda\). Following the proofs in [1], set \(\epsilon\) to be the differential operator \(\epsilon = \lim_{t \to 1} - \frac{d}{dt}\). Then, using the fact that \(\alpha = \lim_{n \to \infty} \eta_n(r) = 0\), by Proposition 2.1, it follows that

\[-\sum_{n=0}^{\infty} (\lambda - \lambda_n)^r = \epsilon(1-t) \sum_{n=0}^{\infty} \eta_n(r) t^n = \epsilon(1-t) \sum_{n=0}^{\infty} (\lambda - \lambda_n)^r t^n\]

\[= \epsilon(1-t) \sum_{n=0}^{\infty} \sum_{k=0}^{r} \binom{r}{k} (\lambda)^{r-k} (\lambda_n)^k t^n\]

\[= \sum_{k=0}^{r} \binom{r}{k} (\lambda)^{r-k} (\lambda_n)^k t^n\]

\[= \sum_{k=1}^{r} \binom{r}{k} (\lambda)^{r-k} (\lambda_n)^k\]

In the fifth line we have used Proposition 2.1 and the fact that if \(k = 0\), then \(\lambda_n^0 = 1\) and \(\epsilon(1-t) \sum_{n \geq 0} t^n = 0\).

3. Proof of (3)

Here we provide proofs of (2) and (3) using Theorem 1. Clearly (2) is just the case \(t = 0\) of (3), so we will just prove, in detail, the identity (3).

Proposition 2 Identity (3) is valid.

Proof. If we take \(r = 2\) in Theorem 1 we get

\[\sum_{n=0}^{\infty} (\alpha - \alpha(n))^2 = 2\alpha \sum_{n=0}^{\infty} (\alpha - \alpha(n)) - \sum_{n=0}^{\infty} (\alpha^2 - \alpha(n)^2) .\]
Now taking \( \alpha := (t)_{\infty}/(q)_{\infty} \) and \( \alpha(n) := (t)_n/(q)_n \), we find \( \alpha \) satisfies (i) and (ii) of Proposition 2.1, and further

\[
\sum_{n=0}^{\infty} \left( \frac{(t)_{\infty}}{(q)_{\infty}} - \frac{(t)_n}{(q)_n} \right)^2 = 2 \frac{(t)_{\infty}}{(q)_{\infty}} \sum_{n=0}^{\infty} \left( \frac{(t)_{\infty}}{(q)_{\infty}} - \frac{(t)_n}{(q)_n} \right)
- \sum_{n=0}^{\infty} \left( \frac{(t)_{\infty}}{(q)_{\infty}} - \frac{(t)_n}{(q)_n} \right)^2 \cdot (q)_{\infty} / (q)_n (1 - q^n) \left( \frac{(q)_n}{(t)_n} + 1 \right). \tag{7}
\]

In [3, Corollary 4.3 (vi)] we find

\[
\sum_{n=0}^{\infty} \left( \frac{(t)_{\infty}}{(q)_{\infty}} - \frac{(t)_n}{(q)_n} \right)^2 = - \frac{(t)_{\infty}}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{(q/t)_n t^n}{(q)_n (1 - q^n)}. \tag{8}
\]

By Corollary 4.2 of [2] with \( a = q \) we have

\[
\sum_{n=0}^{\infty} \left( \frac{(t)_{\infty}}{(q)_{\infty}} - \frac{(t)_n}{(q)_n} \right) = - \frac{(t)_{\infty}}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{(q/t)_n t^n}{(q)_n (1 - q^n)}. \tag{9}
\]

Since \( \alpha = (t)_{\infty}/(q)_{\infty} \), we find by (6) that (3) readily follows after inserting (9) and (8). \( \square \)

4. The Case \( r > 2 \)

The slightly more difficult task with Theorem 1 is finding identities for \( r > 2 \). We consider an identity for general \( r > 2 \), for a particular choice of \( \alpha(n) \). First we need a new result from a paper by Fang [6]. Throughout this section we will use standard notation for the \( q \)-binomial coefficients

\[
\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} := \frac{(1 - q)(1 - q^2) \cdots (1 - q^{m_0})}{(1 - q)(1 - q^2) \cdots (1 - q^{m_1}) (1 - q^2) \cdots (1 - q^{m_0-m_1})}.
\]

**Theorem 3** (Fang [6, Corollary 6.1]) For \( 0 \leq m_{t+1} \leq m_t \leq \cdots \leq m_1 \leq m_0 = m \), where \( t, m \), and the \( m_t \) are non-negative integers, and \( |x| < 1 \), we have

\[
(q; q)_{r+2} \sum_{n=0}^{\infty} \frac{(c; q)_n x^n}{(q; q)_{r+3}} = \frac{(cx; q)_{\infty}}{(x; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(x; q)_m (-1)^m q^{m(m+1)/2}}{(q; cx; q)_m}
\times \sum_{m_1, m_2, \ldots, m_{t+1}} \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \cdots \begin{bmatrix} m_t \\ m_{t+1} \end{bmatrix} q^{\sum_{i=0}^{t+1} m_i} m_{t+1}(m_{t+1}-m_i). \tag{10}
\]
Now set $\alpha := 1/(q)^{t+3}$, $\alpha(n) := 1/(q)^{n+3}$, and put

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_{n+3}}.$$ 

Using Proposition 2.1, Equation (10) with $c = 0$ and $x = z$, we find that

$$\sum_{n=0}^{\infty} \left( 1 - \frac{1}{(q)^{t+3}} \frac{1}{(q)_{n+3}} \right) = \epsilon(1 - z)f(z) = \epsilon(1 - z)\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_{n+3}},$$

$$= \epsilon(1 - z) \frac{1}{(q; q)_{t+2} (z; q)} \sum_{m=0}^{\infty} \frac{(z; q)_m (-1)^m q^{m(m+1)/2}}{(q; q)_m}$$

$$\times \sum_{0 \leq m_{t+1} \leq m_t \leq \ldots \leq m_1 \leq m_0 = m} \left[ \begin{array}{c} m_0 \\ m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_t \\ m_{t+1} \end{array} \right] q^{\sum_{i=0}^{m_{t+1}} m_{i+1}(m_{i+1} - m_i)}.$$ 

$$= \frac{1}{(q)^{t+3} (q; q)_{t+3}} \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} + \frac{1}{(q)^{t+3}} \epsilon \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2} (z; q)_m}{(q; q)_m}$$

$$\times \sum_{0 \leq m_{t+1} \leq m_t \leq \ldots \leq m_1 \leq m_0 = m} \left[ \begin{array}{c} m_0 \\ m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_t \\ m_{t+1} \end{array} \right] q^{\sum_{i=0}^{m_{t+1}} m_{i+1}(m_{i+1} - m_i)}.$$ 

Finally, observing that $\epsilon(z; q)_m/(q; q)_m = -1/(1 - q^m)$, we find the last equation produces the following lemma.

**Lemma 4** For each non-negative integer $t$, we have

$$\sum_{n=0}^{\infty} \left( 1 - \frac{1}{(q)^{t+3}} \frac{1}{(q)_{n+3}} \right) = \frac{1}{(q)^{t+3}} \left( \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} - \sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m+1)/2}}{(1 - q^m)} \right)$$

$$\times \sum_{0 \leq m_{t+1} \leq m_t \leq \ldots \leq m_1 \leq m_0 = m} \left[ \begin{array}{c} m_0 \\ m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_t \\ m_{t+1} \end{array} \right] q^{\sum_{i=0}^{m_{t+1}} m_{i+1}(m_{i+1} - m_i)}.$$ 

We can now obtain the following new result.
Theorem 5 For any integer \( r > 2 \), we have

\[
\sum_{n=0}^{\infty} \left( \frac{1}{(q)_{\infty}} - \frac{1}{(q)_n} \right)^r = r(q)_{\infty} - \frac{r(r-1)(q)_{\infty}^r}{2} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^n(n+1)/2}{1-q^n} \right) \\
\times \left( \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^n(n+1)/2}{1-q^n} \right) \\
- (q)_{\infty}^{r-3} \sum_{k=0}^{r-3} \binom{r}{k+3} (-1)^{k+3} \left( \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} - \sum_{m=1}^{\infty} \frac{(-1)^m q^m(m+1)/2}{1-q^m} \right) \\
\times \sum_{\sum_{i=0}^{k+1} m_i = m} \left[ m_0 \right] \left[ m_1 \right] \left[ m_2 \right] \ldots \left[ m_k \right] q^{\sum_{i=0}^{k+1} m_i + 1 + (m_i + 1 - m_i)}. \tag{12}
\]

Proof. First observe that, from Theorem 1, we can write for \( r > 2 \),

\[
\sum_{n=0}^{\infty} (\alpha - \alpha(n))^r = r\alpha^{r-1} \sum_{n=0}^{\infty} (\alpha - \alpha(n)) - \frac{1}{2} r(r-1)\alpha^{r-2} \sum_{n=0}^{\infty} (\alpha^2 - \alpha(n)^2) \\
- \sum_{k=3}^{r} \binom{r}{k} (\alpha)^{r-k} (-1)^k \sum_{n=0}^{\infty} (\alpha^k - \alpha^k(n)). \tag{13}
\]

Choosing \( \alpha(n) := 1/(q)_n \) and \( \alpha := 1/(q)_{\infty} \) in (13) we obtain

\[
\sum_{n=0}^{\infty} \left( \frac{1}{(q)_{\infty}} - \frac{1}{(q)_n} \right)^r = r(q)_{\infty} - (r-1) \sum_{n=0}^{\infty} \left( \frac{1}{(q)_{\infty}} - \frac{1}{(q)_n} \right) \\
- \frac{r(r-1)(q)_{\infty}^{r-2}}{2} \sum_{n=0}^{\infty} \left( \frac{1}{(q)_{\infty}^2} - \frac{1}{(q)_n^2} \right) \\
- \sum_{k=0}^{r-3} \binom{r}{k+3} (\frac{1}{(q)_{\infty}^r})^{r-(k+3)} (-1)^k \sum_{n=0}^{\infty} \left( (q)_{\infty}^{-(k+3)} - (q)_n^{-(k+3)} \right) . \tag{14}
\]
Since (11) holds for each non-negative integer \( t \), we insert it into the third line of (14) to get

\[
\sum_{n=0}^{\infty} \left( \frac{1}{(q)_n} - \frac{1}{(q)_n} \right)^r = r(q)_\infty \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} - \frac{1}{2} r(r - 1)(q)_\infty
\]

\[
\times \left( \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n(n+1)/2}}{1 - q^n} \right)
\]

\[
- \sum_{k=0}^{r-3} \left( \frac{r}{k+3} \right)(q)_\infty^{r-(k+3)}(-1)^{k+3} \frac{1}{(q)^{k+3}}
\]

\[
 \times \left( \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} - \sum_{m=1}^{\infty} \frac{(-1)^{m}q^{m(m+1)/2}}{1 - q^m} \right)
\]

\[
\times \sum_{0 \leq m_{k+1} \leq m_k \leq \ldots \leq m_1 \leq m_0 = m} \prod_{i=1}^{r} (m_i - m_i(n)), \quad (15)
\]

upon invoking [2, pg.19, eq.(6.7)], and [2, pg.19, eq.(5.4)]. Lastly, canceling out the product \((q)^{-k-3}\), gives the theorem. \( \square \)

5. Concluding Remarks

The results contained herein suggest that there should be some further interesting consequences of Proposition 2.1. One way we could obtain a generalization of Theorem 1 is choosing a different \( \eta_n(r) \) in our proof. Namely, we could choose

\[
\eta_n(r) = \prod_{i=1}^{r} (\alpha_i - \alpha_i(n)),
\]

where at least some of the \( \alpha_i(n) \) are different for each \( i \) with \( 1 \leq i \leq r \). (Choosing all the \( \alpha_i(n) \) to be equal to each other for each \( i \) gives Theorem 1.) The trick here is to ensure that \( \lim_{n \to \infty} \eta_n(r) = 0 \). The difficult task in applications of Theorem 1 to \( q \)-series is finding the simpler, more common expressions for integers \( r > 2 \), for the right hand side of (4). We can, however, obtain more results like Theorem 5 using Fangs’ more general identity [6, pg.1404, Theorem 6.1]. It would be interesting to see some applications of Proposition 2.1 outside of \( q \)-series.
References


