Theorem of prime

Our main divisibility result, Theorem 1, says that, in an appropriate range, this divisibility is the same as that of the coefficients of \((1 \pm \frac{x^n}{p})^t\). Here \(p\) is any prime and \(t\) is any integer, positive or negative. We denote by \(\nu_p(n)\) the exponent of \(p\) in an integer and by \([x^n]f(x)\) the coefficient of \(x^n\) in a power series \(f(x)\).

Theorem 1. If \(t\) is any integer and \(1 \leq m \leq p^{\nu_p(t)}\), then

\[
\nu_p \left( [x^{(p-1)m}]\ell(x)^t \right) = \nu_p(t) - \nu_p(m) - m.
\]

Thus, for example, if \(\nu_3(t) = 2\), then, for \(m = 1, \ldots, 9\), the exponent of 3 in \([x^{2m}]\ell(x)^t\) is, respectively, 1, 0, -2, -3, -5, -5, -6, and -9, which is the same as in \((1 \pm \frac{x}{3})^t\). In Section 3, we will discuss what we can say about \(\nu_p([x^n]\ell(x)^t)\) when \(n\) is not divisible by \((p - 1)\) and \(n < (p - 1)p^\nu_p(t)\).

The motivation for Theorem 1 was provided by ongoing work which seeks to apply the result when \(p = 2\) to make more explicit some nonimmersion results for
complex projective spaces described in [4]. The coefficients studied here can be directly related to Stirling numbers and generalized Bernoulli numbers ([3, Chapter 6]), but it seems that our divisibility results are new in any of these contexts.

Proving Theorem 1 led the author to discover an interesting modification of multinomial coefficients.

**Definition 2.** For an ordered $r$-tuple of nonnegative integers $(i_1, \ldots, i_r)$, not all 0, we define

\[
c(i_1, \ldots, i_r) := \frac{(\sum i_j)! (\sum i_j - 1)!}{i_1! \cdots i_r!}.
\]

Note that $c(i_1, \ldots, i_r)$ equals $(\sum i_j)! / \sum i_j$ times a multinomial coefficient. Surprisingly, these numbers satisfy the same recursive formula as multinomial coefficients.

**Definition 3.** For positive integers $k \leq r$, let $E_k$ denote the ordered $r$-tuple whose only nonzero entry is a 1 in position $k$.

**Proposition 4.** If $I = (i_1, \ldots, i_r)$ is an ordered $r$-tuple of nonnegative integers with $\sum i_j > 1$, then

\[
c(I) = \sum_{i_k > 0} c(I - E_k).
\]  

(1)

If we think of a multinomial coefficient $\left( \frac{\sum i_j}{(i_1)! \cdots (i_r)!} \right) := (i_1 + \cdots + i_r)! / ((i_1)! \cdots (i_r)!)$ as being determined by the unordered $r$-tuple $(i_1, \ldots, i_r)$ of nonnegative integers, then it satisfies the recursive formula analogous to that of (1). For a multinomial coefficient, entries which are 0 can be omitted, but that is not the case for $c(i_1, \ldots, i_r)$.

**Proof of Proposition 4.** The right hand side of (1) equals

\[
\sum_k i_k (\sum i_j - 2)! \binom{\sum i_j - k}{(i_1)! \cdots (i_r)!} \left( \sum_j i_j \right)
\]

\[
= \frac{(\sum i_j - 2)!}{(i_1)! \cdots (i_r)!} \left( (\sum i_k) \left( \sum i_j - \sum i_k k \right) \right)
\]

\[
= \frac{(\sum i_j - 2)!}{(i_1)! \cdots (i_r)!} \left( \sum i_j \right) \left( \sum i_j - 1 \right),
\]

which equals the left hand side of (1).

**Corollary 5.** If $\sum i_j > 0$, then $c(i_1, \ldots, i_r)$ is a positive integer.

**Proof.** Use (1) recursively to express $c(i_1, \ldots, i_r)$ as a sum of various $c(E_k) = k$.  

\[\Box\]
Corollary 6. For any ordered \( r \)-tuple \( (i_1, \ldots, i_r) \) of nonnegative integers and any prime \( p \),
\[
\nu_p \left( \sum i_j \right) \leq \nu_p \left( \sum i_j \right) + \nu_p \left( \sum_{i_1, \ldots, i_r} i_j \right). 
\]

Proof. Multiply numerator and denominator of the definition of \( c(i_1, \ldots, i_r) \) by \( \sum i_j \) and apply Corollary 5.

The proof of Theorem 1 utilizes Corollary 6 and also the following lemma.

Lemma 7. If \( t \) is any integer and \( \sum i_j \leq p^{\nu_p(t)} \), then
\[
\nu_p \left( t - \sum_{i_j, i_1, \ldots, i_r} i_j \right) = \nu_p(t) + \nu_p \left( \sum_{i_1, \ldots, i_r} i_j \right) - \nu_p \left( \sum i_j \right). 
\]

Proof. For any integer \( t \), the multinomial coefficient on the left hand side of (3) equals \( (t - 1) \cdots (t + 1 - \sum i_j)! \), and so the left hand side of (3) equals \( \nu_p(t) - \nu_p(\sum i_j) \). Since \( \nu_p(p - s) = \nu_p(s) \) provided \( 0 < s < p^{\nu_p(t)} \), this becomes \( \nu_p(t) + \nu_p(\sum i_j - 1)! - \nu_p(\sum i_j) \), and this equals the right hand side of (3).

Proof of Theorem 1. By the multinomial theorem,
\[
[x^{p-1}m] t(x)^t = (-1)^{p-1}m \sum T_I, 
\]
where
\[
T_I = \binom{t}{t - \sum_{i_j, i_1, \ldots, i_r} i_j} \frac{1}{\prod (j + 1)^{i_j}}, 
\]
with the sum taken over all \( I = (i_1, \ldots, i_r) \) satisfying \( \sum i_j = (p - 1)m \). Using Lemma 7, we have
\[
\nu_p(T_I) = \nu_p(t) + \nu_p \left( \sum_{i_1, \ldots, i_r} i_j \right) - \nu_p \left( \sum i_j \right) - \sum i_j \nu_p(j + 1). 
\]
If \( I = mE_{p-1} \), then \( \nu_p(T_I) = \nu_p(t) + 0 - \nu_p(m) - m \). The theorem will follow once we show that all other \( I \) with \( \sum i_j = (p - 1)m \) satisfy \( \nu_p(T_I) > \nu_p(t) - \nu_p(m) - m \). Such \( I \) must have \( i_j > 0 \) for some \( j \neq p - 1 \). This is relevant because \( \frac{1}{p - 1} \geq \nu_p(j + 1) \) with equality if and only if \( j = p - 1 \). For \( I \) such as we are considering, we have
\[
\nu_p(T_I) - (\nu_p(t) - \nu_p(m) - m) 
\]
\[
= \nu_p \left( \sum_{i_1, \ldots, i_r} i_j \right) - \sum i_j \nu_p(j + 1) + \nu_p(\sum i_j) + \frac{1}{p - 1} \sum i_j j 
\]
\[
\geq \sum i_j \left( \frac{1}{p - 1} - \nu_p(j + 1) \right) > 0.
\]
We have used (2) in the middle step.
2. Zero Coefficients

While studying coefficients related to Theorem 1, we noticed the following result about occurrences of coefficients of powers of the reciprocal log series which equal 0.

**Theorem 8.** If $m$ is odd and $m > 1$, then $[x^m](\frac{x}{\log(1+x)})^m = 0$, while if $m$ is even and $m > 0$, then $[x^{m+1}](\frac{x}{\log(1+x)})^m = 0$.

Moreover, this property characterizes the reciprocal log series.

**Corollary 9.** A power series $f(x) = 1 + \sum_{i \geq 1} c_i x^i$ with $c_1 \neq 0$ has $[x^m](f(x))^m = 0$ for all odd $m > 1$ and $[x^{m+1}](f(x))^m = 0$ for all even $m > 0$ if and only if $f(x) = \frac{2c_1 x}{\log(1+2c_1 x)}$.

**Proof.** By Theorem 8, the reciprocal log series satisfies the stated property. Now assume that $f$ satisfies this property and let $n$ be a positive integer and $\epsilon = 0$ or $1$. Since

$$[x^{2n+1}](f(x))^{2n+\epsilon} = (2n + \epsilon)(2n + \epsilon - 1)c_1c_{2n} + (2n + \epsilon)c_{2n+1} + P,$$

where $P$ is a polynomial in $c_1, \ldots, c_{2n-1}$, we see that $c_{2n}$ and $c_{2n+1}$ can be determined from the $c_i$ with $i < 2n$.

Our proof of Theorem 8 is an extension of arguments of [1] and [2]. It benefited from ideas of Francis Clarke. The theorem can be derived from results in [3, Chapter 6], but we have not seen it explicitly stated anywhere.

**Proof of Theorem 8.** Let $m > 1$ and

$$\left(\frac{x}{\log(1+x)}\right)^m = \sum_{i \geq 0} a_i x^i.$$

Letting $x = e^y - 1$, we obtain

$$\left(\frac{e^y - 1}{y}\right)^m = \sum_{i \geq 0} a_i (e^y - 1)^i. \quad (6)$$

Let $j$ be a positive integer, and multiply both sides of (6) by $y^m e^y/(e^y - 1)^{j+1}$, obtaining

$$(e^y - 1)^{m-j-1}e^y = y^m \sum_{i \geq 0} a_i (e^y - 1)^{i-j-1}e^y$$

$$= y^m \left( a_j \frac{e^y}{e^y - 1} + \sum_{i \neq j} \frac{a_i}{i-j} \frac{d}{dy} (e^y - 1)^{i-j} \right). \quad (7)$$

Since the derivative of a Laurent series has no \( y^{-1} \)-term, we conclude that the coefficient of \( y^{m-1} \) on the right-hand side of (7) is \( a_j[y^{-1}](1 + \frac{1}{2} y) = a_j \).

The Bernoulli numbers \( B_n \) are defined by \( \frac{y}{e^y - 1} = \sum \frac{B_n}{n!} y^n \). Since \( \frac{y}{e^y - 1} + \frac{1}{2} y \) is an even function of \( y \), we have the well-known result that \( B_n = 0 \) if \( n \) is odd and \( n > 1 \).

Let

\[
j = \begin{cases} 
  m & m \text{ odd} \\
  m + 1 & m \text{ even}.
\end{cases}
\]

For this \( j \), the left-hand side of (7) equals

\[
\left\{ \begin{array}{l}
  1 + \sum \frac{B_n}{n!} y^{n-1} \\
  -\frac{d}{dy}(e^y - 1)^{-1} = -\sum \frac{(i-1)B_i}{i!} y^{i-2}
\end{array} \right.
\]

and comparison with the coefficient of \( y^{m-1} \) in (7) implies

\[
\begin{align*}
  a_m &= \frac{B_m}{m!} = 0 & m \text{ odd} \\
  a_{m+1} &= -\frac{mB_{m+1}}{(m+1)!} = 0 & m \text{ even},
\end{align*}
\]

yielding the theorem. \( \square \)

3. Other Coefficients

In this section, a sequel to Theorem 1, we describe what can be easily said about \( \nu_p([x^{(p-1)m+\Delta}]\ell(x)^t) \) when \( 0 < \Delta < p - 1 \) and \( m < p^{\nu_p(t)} \). This is not relevant in the motivating case, \( p = 2 \). Our first result says that these exponents are at least as large as those of \( [x^{(p-1)m}]\ell(x)^t \). Here \( t \) continues to denote any integer, positive or negative.

**Proposition 10.** If \( 0 < \Delta < p - 1 \) and \( m < p^{\nu_p(t)} \), then

\[
\nu_p\left([x^{(p-1)m+\Delta}]\ell(x)^t\right) \geq \nu_p(t) - \nu_p(m) - m.
\]

**Proof.** We consider terms \( T_I \) as in (4) with \( \sum i_j = (p-1)m + \Delta \). Similarly to (5), we obtain

\[
\nu_p(T_I) - (\nu_p(t) - \nu_p(m) - m) = \nu_p\left(\sum_{i_1, \ldots, i_r} i_j \right) - \nu_p\left(\sum i_j \right) - \sum i_j \nu_p(j+1) + \nu_p(m) + m.
\]

We wish to show that this is nonnegative.
For $I = (i_1, \ldots, i_r)$, let
\[
\tilde{\nu}_p(I) := \nu_p \left( \sum_{i_j} i_j \right) - \nu_p \left( \sum_{i_j} i_j - 1 \right) = \nu_p \left( \frac{1}{\nu_p} \left( \sum_{i_j} i_j \right) \right),
\]
for any $j$. Thus
\[
\tilde{\nu}_p(I) \geq - \min_j \nu_p(i_j).
\]
(9)

Ignoring the term $\nu_p(m)$, the expression (8) is
\[
\geq \tilde{\nu}_p(I) + \sum i_j (\frac{1}{\nu_p(j + 1)} - \nu_p(j + 1)) - \frac{\Delta}{\nu_p}.
\]
(10)

Note that
\[
\sum i_j (\frac{1}{\nu_p(j + 1)} - \nu_p(j + 1)) - \frac{\Delta}{\nu_p} = m - \sum i_j \nu_p(j + 1)
\]
is an integer and is greater than $-1$, and hence is $\geq 0$. Thus we are done if $\tilde{\nu}_p(I) \geq 0$.

Now suppose $\tilde{\nu}_p(I) = -e$ with $e > 0$. By (9), all $i_j$ are divisible by $p^e$. Thus
\[
(p - 1) \sum i_j (\frac{1}{\nu_p(j + 1)} - \nu_p(j + 1))
\]
is a positive integer and divisible by $p^e$. Hence it is $\geq p^e$. Therefore, (10) is an integer which is strictly greater than $-e + \frac{p^e}{\nu_p} - 1 = -e + \sum_{k=1}^{e-1} p^k + \frac{1}{\nu_p}$. Since it is an integer, we can replace the $\frac{1}{\nu_p}$ by 1, and obtain

the nonnegative expression $\sum_{k=0}^{e-1} (p^k - 1)$. We obtain the desired conclusion, that, for each $I$, (10), and hence (8), is $\geq 0$.

Finally, we address the question of when does equality occur in Proposition 10. We give a three-part result, but by the third it becomes clear that obtaining additional results is probably more trouble than it is worth.

**Proposition 11.** In Proposition 10,

(a) the inequality is strict ($\neq$) if $m \equiv 0 (p)$;

(b) equality holds if $\Delta = 1$ and $m \not\equiv 0, 1 (p)$;

(c) if $\Delta = 2$ and $m \not\equiv 0, 2 (p)$, then equality holds if and only if $3m \not\equiv 5 (p)$.

Proof. We begin as in the proof of Proposition 10, and note that, using (2), the value of (8) is greater than or equal to
\[
\nu_p(m) - \frac{\Delta}{p - 1} + \sum i_j (\frac{1}{\nu_p(j + 1)} - \nu_p(j + 1)) - \nu_p((p - 1)m + \Delta).
\]
(11)

(a) If $\nu_p(m) > 0$, then $\nu_p((p - 1)m + \Delta) = 0$ and so (11) is greater than 0.
In (b) and (c), we exclude consideration of the case where \( m \equiv \Delta (p) \) because then \( \nu_p((p-1)m + \Delta) > 0 \) causes complications.

(b) If \( \Delta = 1 \) and \( m \not\equiv 0, 1 (p) \), then for \( I = E_1 + mE_{p-1} \), (8) equals
\[
\nu_p(m+1) - \nu_p(m+1) - m + \nu_p(m) + m = 0,
\]
while for other \( I \), (11) is
\[
0 - \frac{1}{p-1} + \sum i_j \left( \frac{1}{p-1} j - \nu_p(j+1) \right) > 0.
\]

(c) Assume \( \Delta = 2 \) and \( m \not\equiv 0, 2 (p) \). Then
\[
T_{2E_1+mE_{p-1}} + T_{E_2+mE_{p-1}} = \frac{t(t-1) \cdots (t-m+1)}{2^m!} \frac{1}{4p^m} + \frac{t(t-1) \cdots (t-m)}{m!} \frac{1}{3p^m}
\]
\[
= (-1)^m \frac{t}{p^m} \left( \frac{1}{8}(-m-1+A) + \frac{1}{6}(1+B) \right)
\]
\[
= (-1)^m \frac{t}{24p^m} (-3m + 5 + 3A + 8B). \tag{12}
\]

Here \( A \) and \( B \) are rational numbers which are divisible by \( p \). This is true because \( \nu_p(t) > \nu_p(i) \) for all \( i \leq m \). Since \( p > 3 \), (12) has \( p \)-exponent greater than or equal to \( \nu_p(t) - m \), with equality if and only if \( 3m - 5 \not\equiv 0 (p) \). Using (11), the other terms \( T_I \) satisfy
\[
\nu_p(T_I) - (\nu_p(t) - m) \geq \sum i_j \left( \frac{1}{p-1} j - \nu_p(j+1) \right) - \frac{2}{p-1} > 0.
\]

\( \square \)

References


