A NEW APPROACH TO THE RESULTS OF KÖVARI, SÓS, AND TURÁN CONCERNING
RECTANGLE-FREE SUBSETS OF THE GRID

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Abstract
For positive integers \(m\) and \(n\), define \(f(m, n)\) to be the smallest integer such that
any subset \(A\) of the \(m \times n\) integer grid with \(|A| \geq f(m, n)\) contains a rectangle;
that is, there are \(x \in [m]\) and \(y \in [n]\) and \(d_1, d_2 \in \mathbb{Z}^+\) such that all four points
\((x, y), (x + d_1, y), (x, y + d_2),\) and \((x + d_1, y + d_2)\) are contained in \(A\). In 1954,
Kövari, Sós, and Turán showed that \(\lim_{k \to \infty} \frac{f(k, k)}{k^{3/2}} = 1\). They also showed that
\(f(p^2, p^2 + p) = p^2(p + 1) + 1\) whenever \(p\) is a prime number. We recover their
asymptotic result and strengthen the second, providing cleaner proofs which exploit
a connection to projective planes, first noticed by Mendelsohn. We also provide an
explicit lower bound for \(f(k, k)\) which holds for all \(k\).

1. Introduction and Motivation
For a positive integer \(n\), let \([n] = \{1, 2, \ldots, n\}\). For \(m, n \in \mathbb{Z}^+\), define \(f(m, n)\) to be
the least integer such that if \(A \subseteq [m] \times [n]\) with \(|A| \geq f(m, n)\), then \(A\) contains a
rectangle; that is, there is \(x \in [m], y \in [n],\) and \(d_1, d_2 \in \mathbb{Z}^+\) such that all four points
\((x, y), (x + d_1, y), (x, y + d_2),\) and \((x + d_1, y + d_2)\) are contained in \(A\). For ease in
notation, let \(f(k) = f(k, k)\). For \(c \in \mathbb{Z}^+,\) a \(c\)-coloring of a set \(S\) is a surjective map
\(\chi : S \to [c].\) If \(\chi\) is constant on a set \(A \subset S,\) we say that \(A\) is monochromatic.

We will write \(g(k) \sim h(k)\) to mean that functions \(g\) and \(h\) are asymptotically
equal; that is, \(\lim_{k \to \infty} \frac{g(k)}{h(k)} = 1\). Also, notice that \(f(m, n) = f(n, m)\) for any choice of

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n and m.

The problem of finding bounds or exact values of \( f(m, n) \) finds its roots in the famous theorem of van der Waerden from [21], which states that given any positive integers \( c \) and \( d \), there exists an integer \( N \) such that any \( c \)-coloring of \([N]\) contains a monochromatic arithmetic progression of length \( d \). Szemerédi proved a density version of this theorem in [20], using the now well-known Regularity Lemma. Progress in this area is still being made. For instance, in [3], Axenovich and the second author try to find the smallest \( k \) so that in any 2-coloring of \([k] \times [k]\) there is a monochromatic square; i.e., a rectangle with \( d_1 = d_2 \). While the upper bounds are enormous, they proved \( k \geq 13 \); in [4], Bacher and Eliahou show that \( k = 15 \). In [10], the authors are interested in finding \( \text{OBS}_c \), which is the collection of \([m] \times [n]\) grids which cannot be colored in \( c \) colors without a monochromatic rectangle, but every proper subgrid can be; see also [7]. For a more complete survey on van der Waerden type problems, see [11].

Zarankiewicz introduced the problem of finding \( f(m, n) \) in [22] using the language of minors of \((0,1)\)-matrices. In [12], Kövári, Sós, and Turán show that \( f(k) \sim k^{3/2} \) and that whenever \( p \) is a prime number, we have \( f(p^2 + p, p^2) = p^2(p + 1) + 1 \). In this manuscript, we will recover this asymptotic result and strengthen the second result.

In [17], Reiman achieved the bound of

\[
f(m, n) \leq \frac{1}{2} \left( m + \sqrt{m^2 + 4mn(n-1)} \right) + 1.
\]  

Notice that by setting \( m = p^2 + p \) and \( n = p^2 \), the right hand side of (1) becomes \( p^2(p + 1) + 1 \), so the result of Kövári, Sós, and Turán implies that the inequality is sharp. Reiman showed equality in (1) in the case that \( m = n = q^2 + q + 1 \), provided \( q \) is a prime power. In [14], Mendelsohn recovers and strengthens the equality result of Reiman by noticing the connection of the Zarankiewicz problem to projective planes.

A \( k \times k \) \((0,1)\)-matrix \( A \) corresponds to a subset \( S_A \subset [k] \times [k] \) by

\[
(i, j) \in S \text{ if and only if the } (i, j) \text{ entry of } A \text{ is 1.}
\]

Notice that the set \( S_A \) contains a rectangle if and only if the matrix \( A^T A \) has an entry off the main diagonal which is not equal to 0 or 1. Also notice that \( \text{tr}(A^T A) = |S_A| \).

Such \((0,1)\)-matrices arise in the study of projective planes. A projective plane of order \( n \) is an incidence structure consisting of \( n^2 + n + 1 \) points and \( n^2 + n + 1 \) lines such that

(i) any two distinct points lie on exactly one line;

(ii) any two distinct lines intersect in exactly one point;
(iii) each line contains exactly $n + 1$ points; and

(iv) there is a set of 4 points such that no 3 of these points lie on the same line.

It is not known for which positive integers $n$ there exists a projective plane of order $n$; projective planes have been constructed for all prime-power orders, but for no others. In the well-known paper [5], Bruck and Ryser show that if the square-free part of $n$ is divisible by a prime of the form $4k + 3$, and if $n$ is congruent to 1 or 2 modulo 4, then there is no projective plane of order $n$; see also [6]. More recently, the authors in [8] draw a connection between the existence of projective planes of order greater than or equal to 157 and the number of cycles in $n \times n$ bipartite graphs of girth at least 6. In 1989, a computer search conducted by the authors in [13] showed that there is no projective plane of order 10. The smallest order for which it is still not known whether there is a projective plane is 12, although the results in [15, 19, 16, 1, 2] suggest that there is no such structure.

Next we state a lemma which appears in [14] connecting projective planes to the Zarankiewicz problem.

**Lemma 1.** If $n$ is a positive integer such that there exists a projective plane of order $n$, then $f(n^2 + n + 1) = (n + 1)(n^2 + n + 1) + 1$.

We will include a proof of Lemma 1 both for completeness and since we will reference the lower bound construction in the proof of Theorem 4.

**Proof of Lemma 1.** Let $n$ be a positive integer such that there is a projective plane of that order. For ease in notation, set $N = n^2 + n + 1$. First we will show that $f(N) \geq (n + 1)N + 1$.

We begin by constructing a $N \times N$ $(0,1)$-matrix $A$. There exists a projective plane $P$ of order $n$; so let $A$ be the $N \times N$ matrix whose rows correspond to the points of $P$ and whose columns correspond to the lines of $P$ where the $(i,j)$ entry of $A$ is equal to 1 if and only if the point indexed by $i$ lies on the line indexed by $j$. Since any two distinct lines have exactly one point in common, the scalar product of any two distinct columns must be 1; hence, $S_A$ does not contain a rectangle. Since each line contains exactly $(n+1)$ points, $|S_A| = tr(A^TA) = (n+1)N$, so $f(N) \geq (n+1)N + 1$.

Now, suppose $A$ is any $N \times N$ $(0,1)$-matrix with $(n+1)N+1$ nonzero entries, and let $a_i$ denote the number of 1s in row $i$. The number of pairs of 1s in row $i$ is $\binom{a_i}{2}$, so the total number of pairs of 1s from each row is $\sum_{i=1}^{N} \binom{a_i}{2}$. The number of pairs of distinct column indices is $\binom{N}{2}$. If $\sum_{i=1}^{N} \binom{a_i}{2} > \binom{N}{2}$, the pigeonhole principle
implies that there is a pair of column indices such that there are two distinct rows which have 1s in both of those columns; i.e., $S_A$ contains a rectangle.

To see that $\sum_{i=1}^{N} \left( \frac{a_i}{2} \right) > \binom{N}{2}$, recall that the Cauchy-Schwarz inequality gives

$$\left( \sum_{i=1}^{N} a_i \right)^2 \leq \sum_{i=1}^{N} a_i^2 \sum_{i=1}^{N} 1^2.$$  \hfill (2)

Since $\sum_{i=1}^{N} a_i = (n + 1)N + 1$ by assumption, the bound in (2) gives

$$(n + 1)^2 N + 2(n + 1) + \frac{1}{N} \leq \sum_{i=1}^{N} a_i^2.$$  \hfill (3)

Since $\sum_{i=1}^{N} a_i^2 = \sum_{i=1}^{N} a_i(a_i - 1) + \sum_{i=1}^{N} a_i = 2 \sum_{i=1}^{N} \left( \frac{a_i}{2} \right) + (n + 1)N + 1$, inequality (3) gives

$$N \left( (n + 1)^2 - (n + 1) \right) + 2(n + 1) + \frac{1}{N} - 1 \leq 2 \sum_{i=1}^{N} \left( \frac{a_i}{2} \right).$$  \hfill (4)

Since $(n + 1)^2 - (n + 1) = n^2 + n + 1 - 1 = N - 1$, inequality (4) can be rewritten as

$$\frac{N(N - 1)}{2} + n + \frac{1}{N} + \frac{1}{2} \leq \sum_{i=1}^{N} \left( \frac{a_i}{2} \right),$$  \hfill (5)

and since $n > 0$, the left hand side of (5) is bound from below by $\binom{N}{2}$, as desired. \hfill \Box

It is interesting to note that we have equality in (2) just in case all of the $a_i$ are equal; that is, each row and column contain the same number of 1s.

2. Main Results

Our main lemma is below, a useful proposition for dealing with asymptotic behavior of functions when some explicit values of the functions are known. A version of this lemma is used in [12], but it is neither proved nor explicitly stated.

**Lemma 2.** Suppose $g$ and $h$ are monotonically increasing functions. If $a_n$ is a strictly increasing sequence of positive integers such that
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(i) \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1; \)

(ii) \( \lim_{n \to \infty} \frac{h(a_{n+1})}{h(a_n)} = 1; \) and

(iii) \( g(a_n) = h(a_n) \) for all \( n, \)

all hold, then \( g \sim h. \)

Theorem 3 recovers the asymptotic result of Kővari, Sós, and Turán. Theorem 4 strengthens another of their results. The proofs exploit the connection to projective planes, cleaning up the arguments found in [12]. Theorem 5 is an explicit lower bound for \( f(k), \) which holds for all \( k. \)

**Theorem 3.** \( f(k) \sim k^{3/2}. \)

**Theorem 4.** Let \( n \) be a positive integer. If there is a projective plane of order \( n, \)
then \( f(n^2, n^2 + n) = n^2 (n + 1) + 1. \)

**Theorem 5.** If \( k \in \mathbb{Z} \) with \( k \geq 3, \)
then \( f(k) \geq \frac{1}{16} ((k + 4)\sqrt{4k - 3} + 5k + 22). \)

3. **Proof of Lemma 2**

Now we prove Lemma 2.

**Proof.** Let \( g \) and \( h \) be monotonically increasing functions. Suppose \( a_n \) is a strictly increasing sequence of positive integers such that \( \lim_{n \to \infty} \frac{h(a_{n+1})}{h(a_n)} = 1 \) and that \( g(a_n) = h(a_n) \) for all \( n. \) Let \( \varepsilon > 0. \) Choose \( N \) so that

\[
\left| \frac{h(a_{n+1})}{h(a_n)} - 1 \right| < \varepsilon \quad \text{and} \quad \left| \frac{h(a_n)}{h(a_{n+1})} - 1 \right| < \varepsilon \quad (6)
\]

whenever \( n > N. \) Next, choose \( m \) large enough so that for some \( n > N, \) we have \( a_n \leq m \leq a_{n+1}. \) Since \( g \) is increasing and \( g \) and \( h \) agree on the sequence \( a_n, \) we have

\[
h(a_n) = g(a_n) \leq g(m) \leq g(a_{n+1}) = h(a_n + 1). \quad (7)
\]

Since \( h \) is monotone increasing, \( h(a_n) \leq h(m) \leq h(a_{n+1}), \) so we may transform (7) into

\[
\frac{h(a_n)}{h(a_{n+1})} \leq \frac{g(m)}{h(m)} \leq \frac{h(a_{n+1})}{h(a_n)}. \quad (8)
\]

Subtracting 1 from every term in (8) and taking absolute values gives that either

\[
\left| \frac{g(m)}{h(m)} - 1 \right| \leq \left| \frac{h(a_{n+1})}{h(a_n)} - 1 \right| \quad \text{or} \quad \left| \frac{g(m)}{h(m)} - 1 \right| \leq \left| \frac{h(a_n)}{h(a_{n+1})} - 1 \right|.
\]
Without loss of generality, say \(|g(m) h(m)^{-1} - 1| \leq |h(a_{n+1}) h(a_n)^{-1} - 1|\). By (6), we have

\[
\left| \frac{g(m)}{h(m)} - 1 \right| < \varepsilon,
\]

so \(\frac{g}{h} \rightarrow 1\) and \(g \sim h\), as desired. \(\square\)

4. Proof of Theorem 3

Now we prove Theorem 3.

Proof. For a positive integer \(k\), set

\[
h(k) = \left( \sqrt{k - \frac{3}{4} + \frac{1}{2}} \right) k + 1.
\]

Notice that \(h(k) \sim k^{3/2}\) and that \(h(n^2 + n + 1) = (n + 1)(n^2 + n + 1) + 1\), so by Lemma 1, we have \(f(n^2 + n + 1) = h(n^2 + n + 1)\) whenever there is a projective plane of order \(n\). Since there a projective plane of order \(p\) for every prime \(p\), we have that \(f\) and \(h\) agree on an infinite sequence of integers \(a_n\) for which \(\frac{a_n+1}{a_n} \rightarrow 1\) (see [18, 9]). Notice that \(\frac{h(a_{n+1})}{h(a_n)} \rightarrow 1\), so we may apply Lemma 2 to achieve \(f \sim h\), and thus \(f \sim k^{3/2}\), as desired. \(\square\)

5. Proof of Theorem 4

Proof. Let \(n\) be a positive integer such that there is a projective plane of order \(n\). Set \(N = n^2 + n + 1\). As in the proof of Lemma 1, we can construct an \(N \times N\) matrix \(A\) such that \(tr\ (A^TA) = (n + 1)N\) and that \(A^TA\) has only 1s off the main diagonal; hence, the corresponding subset \(S_A\) of the \(N \times N\) grid has no rectangle.

To construct an \(n^2 \times (n^2 + n)\) matrix \(B\) from \(A\), we delete the first column of \(A\) along with all rows having a 1 in the first column. Since each row and column of \(A\) contains exactly \(n + 1\) nonzero entries, we have deleted \(n + 1\) rows and 1 column. The resulting matrix \(B\) is thus an \(n^2 \times (n^2 + n)\) matrix. Since \(A^TA\) has no entries off the main diagonal greater than 1, \(B^TB\) has no entries off the main diagonal greater than 1. Since we have deleted \((n + 1)^2\) nonzero entries from \(A\), we have that

\[
|S_B| = (n + 1)N - (n + 1)^2 = (n + 1)(n^2 + n + 1) - (n + 1)^2 = n^2(n + 1),
\]
so \( f(n^2, n^2 + n) \geq n^2(n + 1) + 1. \)

Using the inequality from Reiman (1),

\[
f(n^2, n^2 + n) \leq n^2(n + 1) + 1,
\]

and hence \( f(n^2, n^2 + n) = n^2(n + 1) + 1, \) as desired.

The structure obtained by taking a projective plane and deleting a line together with all of the points on that line is called an affine plane. Our result is stronger than that of the authors in [12], since we need only that there is a projective plane of order \( n \), not that \( n \) is a prime number.

6. Proof of Theorem 5

Proof. Suppose \( k \) is an integer with \( k \geq 3 \). There exists a nonnegative integer \( \alpha \) such that

\[
2^{2\alpha} + 2^\alpha + 1 \leq k \leq 2^{2\alpha+2} + 2^{\alpha+1} + 1. \tag{9}
\]

By focusing on the upper bound from (9), this gives \( k \leq (2^{\alpha+1} + 1/2)^2 + 3/4, \) or

\[
\frac{\sqrt{k - 3/4} - 1/2}{2} \leq 2^\alpha. \tag{10}
\]

Let \( g(n) = (n + 1)(n^2 + n + 1) + 1, \) and let \( h(k) = \frac{\sqrt{k - 3/4} - 1/2}{2}. \) Since \( g \) is an increasing function, inequality (10) gives

\[
g(h(k)) \leq g(2^\alpha). \tag{11}
\]

By Lemma 1, we have \( g(n) = f(n^2 + n + 1) \) whenever there exists a projective plane of order \( n \). Since there is a projective plane of any prime power order, (11) gives

\[
g(h(k)) \leq f(2^{2\alpha} + 2^{\alpha} + 1). \tag{12}
\]

But since \( f \) is increasing, the lower bound in (9) gives \( g(h(k)) \leq f(k), \) and since \( g(h(k)) = \frac{1}{16} ((k + 4)\sqrt{4k - 3} + 5k + 22) \), we have the desired result.

We also note that while \( g(h(k)) \sim \frac{1}{8} k^{3/2} \), which is worse than the result in Theorem 3, this lower bound holds for every choice of \( k \), and not just those \( k \) for which there exists a projective plane of order \( k \).
7. Further Research

Trying to find the exact value of \( f(m, n) \) without conditions on \( m \) and \( n \) (that is, removing the extra hypotheses from the results in [12]) would be attractive, although this problem has been open for years, and likely requires a new idea.

The next attractive direction is to take the approach of the authors in [10], and consider colorings of rectangular grids.

Recall that \( \text{OBS}_c \) is the collection of \( [m] \times [n] \) grids which cannot be colored in \( c \) colors without a monochromatic rectangle, but every proper subgrid can be. An open problem from [10] is the rectangle-free conjecture: if there exists a rectangle-free subset of \( [m] \times [n] \) of size \( \lfloor mn/c \rfloor \), then it is possible to color \( [m] \times [n] \) in \( c \) colors so there is no monochromatic rectangle. Since the authors in [10] have theorems which depend on the rectangle-free conjecture, resolving this conjecture either in the affirmative or the negative would result in progress for obtaining \( |\text{OBS}_c| \) or even \( \text{OBS}_c \).

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References


