BRUN MEETS SELMER

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Abstract
The most famous 2-dimensional continued fraction algorithm is the Jacobi algorithm. However, Brun and Selmer algorithms are also interesting 2-dimensional subtractive algorithms. Schratzerger shows that all these three algorithms are deeply related by a process similar to insertion and extension for continued fractions. In this note the basic ergodic properties of two mixtures of both maps are explored. Furthermore a digression to a quite different map is made which exhibits an “exotic” invariant measure.

1. Introduction
The most famous 2-dimensional continued fraction algorithm is the Jacobi algorithm. Last years saw an increasing interest in other 2-dimensional algorithms (see [9], chapters 6 and 7, and [2]). The Brun and the Selmer algorithms are remarkable examples of this type. In the first section we give a short description of both algorithms and look shortly on the flip-flop map built on both maps. It generalizes the 1-dimensional map

\[
x \mapsto \frac{x}{1 - x}, \quad 0 \leq x \leq \frac{1}{2}
\]

\[
x \mapsto \frac{1 - x}{x}, \quad \frac{1}{2} \leq x \leq 1
\]

to the set

\[B := \{(x_1, x_2) : \quad 0 \leq x_2 \leq x_1 \leq 1\}.
\]

The jump map (see [9], chapter 3) which avoids the critical point (0,0) leads to Garrity’s triangle sequence (Assaf et al. [1]). The next section is devoted to the study of the composition of the Brun and the Selmer map. The set

\[D^- := \{(x_1, x_2) \in B : x_1 + x_2 \leq 1\}
\]
is transient for the Selmer map and therefore the study of its ergodic behaviour concentrates on the set

\[ D^+ := \{(x_1, x_2) \in B : x_1 + x_2 \geq 1\} . \]

The Brun map expands this set \( D^+ \) onto the full set \( B \). Therefore, the study of the interplay of these different dynamics may be of some interest. In the last section a digression to a different map is made which exhibits an “exotic” invariant measure. “Exotic” means that it is possible to construct a fractal like set with positive Lebesgue measure and an invariant density.

2. The Brun, the Selmer Algorithm, and the Flip-flop Map

The Brun algorithm \( T : B \to B \) is given by the matrices of its inverse branches

\[
M_\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_\gamma = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

which correspond to a partition of \( B \) into three cells \( B(\alpha) = M_\alpha B, B(\beta) = M_\beta B, \)
and \( B(\gamma) = M_\gamma B = D^+ \) (see Figure 1).

The Selmer algorithm \( S : B \to B \) is defined by the matrices of its inverse branches

\[
M_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

There is an important difference to be observed. \( M_0 B \) is the triangle \( B(0) = D^- \) with vertices \([1, 0, 0], [1, 1, 0] \) and \([2, 1, 1] \) but \( M_1 \) and \( M_2 \) are restricted to the triangle \( D^+ \). Then \( M_1 D^+ \) is the triangle \( B(1) \) with vertices \([1, 1, 0], [2, 2, 1] \) and \([2, 1, 1] \). \( M_2 D^+ \) is the triangle \( B(2) \) with vertices \([1, 1, 1], [2, 2, 1] \) and \([2, 1, 1] \) (see Figure 2).

The flip-flop map uses the matrices \( M_0 \) and \( M_\gamma \). It gives the (forward) map

\[
F(x_1, x_2) = \left( \frac{x_1}{1-x_2}, \frac{x_2}{1-x_2} \right); \quad x \in B(0),
\]

\[
F(x_1, x_2) = \left( \frac{x_2}{x_1}, \frac{1-x_1}{x_1} \right); \quad x \in B(\gamma).
\]

Although Pipping used a kind of mixture of both algorithms [6] this kind of a flip-flop between both algorithms seems not to be investigated. We show that this algorithm admits a \( \sigma \)-finite invariant measure but is related to Garrity’s triangle sequence.
A product of $n$ matrices $M_{ij}$, $\eta \in \{0, \gamma\}$ gives a matrix $((B_{ij}^{(n)}))$, $0 \leq i, j \leq 2$ and the Jacobian of an inverse branch after $n$ steps is given by

$$\omega(\eta_1, \ldots, \eta_n; x) = \frac{1}{(B_{00}^{(n)} + B_{01}^{(n)} x_1 + B_{02}^{(n)} x_2)^2}.$$ 

Therefore the measure of a cylinder of rank $n$ is given by

$$\lambda(B(\eta_1, \ldots, \eta_n)) = \frac{1}{2B_{00}^{(n)}(B_{00}^{(n)} + B_{01}^{(n)})(B_{00}^{(n)} + B_{01}^{(n)} + B_{02}^{(n)}).}$$

**Theorem 1:** The function

$$h(x_1, x_2) = \frac{1}{x_1 x_2}$$

is the density of a $\sigma$-finite invariant measure.

This assertion is easily verified.

If we consider the jump map over the cylinder $B(0)$ we obtain a map with matrices

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & k & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

This algorithm is Garrity’s triangle sequence (see e. g. [1, 4, 10]). Therefore the map $F$ is ergodic. Since the segment $(0, 0)(1, 0)$ is pointwise invariant it is no surprise that this algorithm does not converge everywhere. If $p^{(s)} = p(k_1, \ldots, k_s)$ and $q^{(s)} = q(k_1, \ldots, k_s)$ are the vertices of the cylinder $B(k_1, \ldots, k_s)$ such that $F^s p^{(s)} = (0, 0)$ and $F^s q^{(s)} = (1, 0)$ then the segments $p(k_1, \ldots, k_s, k_{s+1}), q(k_1, \ldots, k_s, k_{s+1})$ converge to the segment $p(k_1, \ldots, k_s), q(k_1, \ldots, k_s)$ as $k_{s+1} \to \infty$. Then we choose a sequence $(k_1, k_2, k_3, \ldots)$ such that

$$\frac{d(p(k_1, \ldots, k_s, k_{s+1}), q(k_1, \ldots, k_s, k_{s+1}))}{d(p(k_1, \ldots, k_s), q(k_1, \ldots, k_s))} = \frac{k_s}{1 + k_s}$$

and the infinite product $\prod_s \frac{k_s}{1 + k_s}$ converges. More details can be found in Assaf et al. [1].

**3. The Composition of Both Maps**

We now consider the mixed map $(S \circ T)x = T(Sx)$. Since $SB(1) = SB(2) = D^+ = B(\gamma)$ the map $S \circ T$ can be described by the five matrices

$$M_{00} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_{01} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_{02} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, M_{03} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{04} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
These five matrices give a partition of \( B \) into five cylinders (see Figure 3).

**Lemma 1:** The set

\[
E = \{x : (S \circ T)^j x \in B(0\alpha) \cup B(0\beta) \text{ for all } j \geq 0\}
\]

has measure \( \lambda(E) = 0 \).

**Proof.** The product of \( N \) matrices \( M_{0\alpha} \) and \( M_{0\beta} \) has the form

\[
M^{(N)} = \begin{pmatrix}
B_{00}^{(N)} & B_{01}^{(N)} & B_{02}^{(N)} \\
B_{10}^{(N)} & B_{11}^{(N)} & B_{12}^{(N)} \\
0 & 0 & 1
\end{pmatrix}.
\]

Therefore \( x = (x_1, x_2) \) is mapped onto

\[
(x_1^{(N)}, x_2^{(N)}) = \left( \frac{B_{10}^{(N)} + B_{11}^{(N)} x_1 + B_{12}^{(N)} x_2}{B_{00}^{(N)} + B_{01}^{(N)} x_1 + B_{02}^{(N)} x_2}, \frac{x_2}{B_{00}^{(N)} + B_{01}^{(N)} x_1 + B_{02}^{(N)} x_2} \right).
\]

This implies \( \lim_{N \to \infty} x_2^{(N)} = 0 \). \( \square \)

**Lemma 2:** We have \( B_{02}^{(N)} \leq B_{00}^{(N)} + B_{01}^{(N)} \).

**Proof.** For \( N = 1 \) this is verified by inspection. Then we use induction. Let \( 0\alpha \) or \( 0\beta \) be the \( N \)-th digit. Then

\[
B_{02}^{(N+1)} = B_{02}^{(N)} \leq B_{00}^{(N)} + B_{02}^{(N)} \leq B_{00}^{(N)} + B_{01}^{(N)} = B_{00}^{(N+1)} + B_{01}^{(N+1)}.
\]

If \( \epsilon_N \in \{0\gamma, 1\gamma, 2\gamma\} \) the assertion is immediate.

Now we consider the jump transformation \( R : B \to B \) which leaves out the digits \( 0\alpha \) and \( 0\beta \). This means we define

\[
Rx := (S \circ T)^n x
\]

if \( x \in B(\epsilon_1, \ldots, \epsilon_n), \ \epsilon_1, \ldots, \epsilon_{n-1} \in \{0\alpha, 0\beta\} \) but \( \epsilon_n \in \{0\gamma, 1\gamma, 2\gamma\} \). Lemma 1 implies that \( R \) is defined almost everywhere. \( \square \)

**Lemma 3:** \( R \) satisfies a Rényi condition.
Proof. Let
\[
\omega(\varepsilon_1, \ldots, \varepsilon_N; x) = \frac{1}{(B_{00}^{(N)} + B_{01}^{(N)} x_1 + B_{02}^{(N)} x_2)^3}
\]
be the Jacobian of an inverse branch of \( R \). We have to compare \( B_{00}^{(N)} \) with \( B_{00}^{(N)} + B_{01}^{(N)} + B_{02}^{(N)} \). Since \( \varepsilon_N \in \{0\gamma, 1\gamma, 2\gamma\} \) we see that
\[
B_{00}^{(N)} \geq B_{00}^{(N-1)} + B_{01}^{(N-1)}
\]
but
\[
B_{00}^{(N)} + B_{01}^{(N)} + B_{02}^{(N)} \leq 3B_{00}^{(N-1)} + 2B_{01}^{(N-1)} + B_{02}^{(N-1)} \leq 4B_{00}^{(N-1)} + 3B_{01}^{(N-1)}
\]
by Lemma 2.

\[ \square \]

Lemma 4: If the sequence \( (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) \) contains one of the digits \( 0\gamma, 1\gamma, \) or \( 2\gamma \) infinitely often then
\[ \lim_{n \to \infty} \text{diam } B(\varepsilon_1, \ldots, \varepsilon_n) = 0. \]

Proof. We describe the vertices of the cylinders we consider as the pictures of points in projective coordinates (see Figure 4) and suppress the upper index of the relevant matrix
\[
\beta = \beta(\varepsilon_1, \ldots, \varepsilon_N) = \begin{pmatrix}
B_{00} & B_{01} & B_{02} \\
B_{10} & B_{11} & B_{12} \\
B_{20} & B_{21} & B_{22}
\end{pmatrix}.
\]

We look for triangles which lie inside the triangle \( B(\varepsilon_1, \ldots, \varepsilon_N) \) and contain the triangle \( B(\varepsilon_1, \ldots, \varepsilon_N, \varepsilon_{n+1}) \) or in some cases the triangle \( B(\varepsilon_1, \ldots, \varepsilon_N, \varepsilon_{n+1}, \varepsilon_{n+2}) \). If the points \([a, b, c], [a', b', c'], [a'', b'', c'']\] are collinear such that
\[
\lambda[a, b, c] + [a', b', c'] = [a'', b'', c'']
\]
we will estimate the ratio
\[
\frac{d([\beta[a, b, c], \beta[a', b', c']])}{d([\beta[a, b, c], \beta[a'', b'', c'']])} = \frac{B_{00} a'' + B_{01} b'' + B_{02} c''}{B_{00} a' + B_{01} b' + B_{02} c'}.
\]

We further use that for \( \alpha < \delta \) the function \( f(t) = \frac{t + \alpha}{t + \delta} \) is increasing on \( 0 \leq t \).

\[
\varepsilon_{n+1} = 00
\]

\[
\frac{d([\beta[1, 0, 0], \beta[2, 1, 0]])}{d([\beta[1, 0, 0], \beta[1, 1, 0]])} = \frac{B_{00} + B_{01}}{2B_{00} + B_{01}}.
\]

\[
\frac{d([\beta[1, 0, 0], \beta[3, 1, 1]])}{d([\beta[1, 0, 0], \beta[1, 1, 1]])} = \frac{B_{00} + B_{01} + B_{02}}{3B_{00} + B_{01} + B_{02}} \leq \frac{B_{00} + B_{01}}{2B_{00} + B_{01}}.
\]
Since the periodic point $\overline{\text{w}}$ shrinks to the point $(0, 0)$ we can additionally assume that $\varepsilon_n \in \{0, 3\gamma, 1\gamma, 2\gamma\}$. Then the recursion relations show $B_{00} \leq 2B_{00}$ and we obtain
\[
\frac{B_{00} + B_{01}}{2B_{00} + B_{01}} \leq \frac{3}{4}.
\]

$\varepsilon_{n+1} = 2\gamma$

In a similar way as before we find the ratios
\[
\frac{d(\beta[1, 1, 1], \beta[2, 2, 1])}{d(\beta[1, 1, 1], \beta[1, 1, 0])} = \frac{B_{00} + B_{01}}{2B_{00} + 2B_{01} + B_{02}} \leq \frac{1}{2},
\]
\[
\frac{d(\beta[1, 1, 1], \beta[2, 1, 1])}{d(\beta[1, 1, 1], \beta[1, 0, 0])} = \frac{B_{00}}{2B_{00} + B_{01} + B_{02}} \leq \frac{1}{2}.
\]

$\varepsilon_{n+1} = 0\beta$

Here we use the additional points $\beta[3, 2, 1]$ and $\beta[2, 1, 1]$ which lie outside on the line which joins $\beta[1, 1, 0]$ and $\beta[2, 1, 1]$.

$\varepsilon_{n+2} = 0\beta, 0\gamma, 1\gamma$

\[
\frac{d(\beta[2, 1, 0], \beta[3, 2, 0])}{d(\beta[2, 1, 0], \beta[1, 1, 0])} = \frac{B_{00} + B_{01}}{3B_{00} + 2B_{01}} \leq \frac{1}{2},
\]
\[
\frac{d(\beta[2, 1, 0], \beta[5, 3, 1])}{d(\beta[2, 1, 0], \beta[3, 2, 1])} = \frac{3B_{00} + 2B_{01} + B_{02}}{5B_{00} + 3B_{01} + B_{02}} \leq \frac{3}{4},
\]
\[
\frac{d(\beta[2, 1, 0], \beta[4, 2, 1])}{d(\beta[2, 1, 0], \beta[2, 1, 1])} = \frac{2B_{00} + B_{01} + B_{02}}{4B_{00} + 2B_{01} + B_{02}} \leq \frac{2}{3},
\]
\[
\frac{d(\beta[2, 1, 0], \beta[5, 2, 1])}{d(\beta[2, 1, 0], \beta[3, 1, 1])} = \frac{3B_{00} + B_{01} + B_{02}}{5B_{00} + 2B_{01} + B_{02}} \leq \frac{2}{3}.
\]

$\varepsilon_{n+1} = 0\gamma$

Here we use the additional points $\beta[3, 2, 0], [2, 1, 0]$, and $\beta[1, 0, 0]$.

$\varepsilon_{n+2} = 0\gamma, 1\gamma$

\[
\frac{d(\beta[2, 1, 1], \beta[5, 3, 1])}{d(\beta[2, 1, 1], \beta[3, 2, 0])} = \frac{3B_{00} + 2B_{01}}{5B_{00} + 3B_{01} + B_{02}} \leq \frac{2}{3},
\]
\[
\frac{d(\beta[2, 1, 1], \beta[4, 2, 1])}{d(\beta[2, 1, 1], \beta[2, 1, 0])} = \frac{2B_{00} + B_{01}}{4B_{00} + 2B_{01} + B_{02}} \leq \frac{1}{2}.
\]
\[ \frac{d(\beta[2, 1, 1], \beta[5, 2, 2])}{d(\beta[2, 1, 1], \beta[1, 0, 0])} = \frac{B_{00}}{5B_{00} + 2B_{01} + 2B_{02}} \leq \frac{1}{5}. \]

Only the case \( \overline{T} \gamma \) remains; however, the sequence of associated triangles shrinks to the point \((\lambda - 1, \lambda^2 - \lambda - 1)\), where \( \lambda > 1 \) is the greatest root of \( \lambda^3 = \lambda^2 + 2\lambda - 1 \). \( \Box \)

Lemmas 1-4 provide the necessary machinery to deduce the following:

**Theorem 2:** \( S \circ T \) is ergodic and admits a \( \sigma \)-finite invariant measure \( \mu \sim \lambda \).

**Remark:** The map \((T \circ S)(x) = S(Tx)\) divides \( B \) into nine cells. Since \( S \circ (T \circ S) = (S \circ T) \circ S \) their ergodic behaviors are equivalent.

4. **A Split Algorithm**

The next algorithm is not directly related to the Brun or the Selmer algorithm but shows that the “exotic” behaviour which was first detected with the Parry-Daniels map is quite common (see [5]).

The starting point are the three matrices

\[ \beta(1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \beta(2) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \beta(3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \]

These matrices form a 2-dimensional continued fraction on the basic set \((\mathbb{R}^+)^2\) with the three inverse branches

\[ V(1)(u, v) = \frac{1 + u}{1 + v}, V(2)(u, v) = \frac{1 + v}{1 + u}, V(3)(u, v) = \frac{1}{1 + v}. \]

and the basic partition is

\[ B(1) = \{(u, v) : 1 \leq u, 1 \leq v\}, \quad B(2) = \{(u, v) : 0 \leq v \leq u, v \leq 1\}, \quad B(3) = \{(u, v) : 0 \leq u \leq v, u \leq 1\}. \]

The dual map is given given as

\[ V^\#(1)(x, y) = \left( \frac{x}{1 + x + y}, \frac{y}{1 + x + y} \right). \]
\[ V^\#(2)(x, y) = \left( \frac{x}{1 + x + y}, \frac{1}{1 + x + y} \right) \]
\[ V^\#(3)(x, y) = \left( \frac{1}{1 + x + y}, \frac{y}{1 + x + y} \right) \]

which may be compared with the 2-dimensional Farey-Brocot algorithm which was considered in Schweiger [10]. This algorithm sits on a set \( E \) with \( \lambda(E) = 0 \) but the function
\[ g(x_1, x_2) = \frac{1}{x_1 x_2} \]
behaves formally as an invariant density. It would be nice to explore if in some limiting sense the integral
\[ \int_E \frac{d x_1 d x_2}{x_1 x_2} \]
is finite.

Let
\[ E_{12} = \{ (u, v) : T^s(u, v) \in B(1) \cup B(2), s \geq 0 \} \]
and
\[ E_{13} = \{ (u, v) : T^s(u, v) \in B(1) \cup B(3), s \geq 0 \} \].

We will show that \( \lambda(E_{12}) = \lambda(E_{13}) > 0 \) and calculate an invariant density for the map \( T \) restricted to \( E_{12} \).

We consider the first return map on the set on the set \( B(2) \) of the restriction of \( T \) to \( E_{12} \). This map is given as \( R(u, v) = T^k(u, v) \) if \( (u, v) \in B(2), T^j(u, v) \in B(1), 1 \leq j \leq k - 1, T^k(u, v) \in B(2) \). The associated matrices are given as
\[ \beta(2) \beta(1)^k : \gamma(a) = \left( \begin{array}{ccc} a & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \]
where \( a = k + 1 \). These matrices are related to continued fractions! If
\[ \gamma(a_1) \ldots \gamma(a_s) = \left( \begin{array}{ccc} q_s & 0 & q_{s-1} \\ r_s & 1 & r_{s-1} \\ p_s & 0 & p_{s-1} \end{array} \right) \]
then as usual \( q_s = a_s q_{s-1} + q_{s-2}, p_s = a_s p_{s-1} + p_{s-2} \) but \( r_s = a_s r_{s-1} + r_{s-2} + a_s \).

The last recursion can be written as \( r_s + 1 = a_s (r_{s-1} + 1) + r_{s-2} + 1 \) which shows that \( q_s \leq r_s \leq 2q_s \).

**Theorem 3:** \( \lambda(E_{12}) > 0 \).

**Proof.** We transport the map \( T \) into the triangle with vertices \((0, 0), (1, 0), \) and \((0, 1)\) by using the map \( \psi(u, v) = \left( \frac{u}{1+u+v}, \frac{v}{1+u+v} \right) \). The quotient of the measure of the cylinder \( B(a_1, \ldots, a_s) \) and the length of the associated continued fraction interval \( I(a_1, \ldots, a_s) \) is bounded from below. Therefore we find \( \lambda(E_{12}) > 0 \). \( \square \)
**Theorem 4:** Let $\theta = [a_1, a_2, \ldots]$ be a regular continued fraction and define $\Gamma(\theta) = \sum_{n=0}^{\infty} (\prod_{j=0}^{n} T_j \theta) a_{n+1}$. Then the function

$$ h(u, v) = \frac{1}{(1 + v)(u - \Gamma(v))} $$

is an invariant density for the map $T$ restricted to the set $E_{12}$.

**Proof.** We first remark

$$ \Gamma(\theta) = \Gamma\left(\frac{1}{a + \theta}\right)(a + \theta) - a. $$

Then we calculate

$$ \sum_{a=1}^{\infty} h\left(\frac{a + u}{a + v}, \frac{1}{a + v}\right) \frac{1}{(a + v)^3} = \sum_{a=1}^{\infty} \frac{1}{(a + 1 + v)(a + v)(a + u - \Gamma((a + v)\theta)\Gamma(v))} $$

$$ = \frac{1}{u - \Gamma(v)} \sum_{a=1}^{\infty} \frac{1}{(a + v)(a + 1 + v)} = \frac{1}{(1 + v)(u - \Gamma(v))}. $$

**Remark:** The dual map defined by

$$ \gamma^\#(a) = \begin{pmatrix} a & a & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} $$

formally has the invariant density

$$ f(x_1, x_2) = \frac{1}{x_1} \int_{0}^{1} \frac{dv}{(1 + x_1 \Gamma(v) + x_2 v)^2}. $$

We verify this by direct calculation:

$$ \sum_{a=1}^{\infty} f\left(\frac{x_1}{a + ax_1 + x_2}, \frac{1}{a + ax_1 + x_2}\right) \frac{1}{(a + ax_1 + x_2)^3} $$

$$ = \frac{1}{x_1} \sum_{a=1}^{\infty} \int_{0}^{1} \frac{dv}{(a + (a + \Gamma(v))x_1 + x_2 + v)^2} $$

$$ = \frac{1}{x_1} \sum_{a=1}^{\infty} \int_{\frac{1}{x_1}}^{\frac{1}{x_1}} \frac{d\bar{w}}{(1 + \Gamma(w) x_1 + x_2 \bar{w})^2} = F(x_1, x_2). $$

This follows from $w = \frac{1}{a + v}$ and the equation $\Gamma(v) + a = \Gamma(w)(a + v)$.

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References


Figure 1

Figure 2