A SHORT PROOF OF A RESULT OF GICA AND LUCA

Mickey Polasub
Nuanchan, Bewungkum, Bangkok, Thailand
mickey.polasub@gmail.com

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Abstract
We provide a short proof of a generalization of a recent result of Gica and Luca on the diophantine equation $2^x = y^2 + z^2(x^2 - 2)$.

- Dedicated to the memory of Preechar Polasub.

1. Introduction
In a recent paper, Gica and Luca [2], in connection with a result of Lee [3] on the class number one problem for quadratic fields of the shape $\mathbb{Q}(\sqrt{n^2 + 2})$, were led to consider the diophantine equation

$$2^x = y^2 + z^2(x^2 - 2),$$

(1)

where $x, y$ and $z$ are positive integers. This equation has many known solutions, including $(x, y, z) = (3, 1, 1), (5, 3, 1), (7, 9, 1)$, and infinite families of solutions obtained by choosing, say,

$$x = 2^a + 1, \quad y = 2^{2^{a-1}} (2^a - 1), \quad z = 2^{2^{a-1} - 1}$$

or

$$x = 2^a - 1, \quad y = 2^{2^{a-1}} (2^a + 1), \quad z = 2^{2^{a-1} - 1}.$$  

Gica and Luca prove that there are only the three known solutions with $z = 1$. Their argument relies fundamentally upon lower bounds for linear forms in $p$-adic logarithms. Our goal in this short note is to give a quick proof of a stronger result, which immediately generalizes to partially resolve a conjecture of Gica and Luca on equation (1). Apparently, this strengthening does not follow from the techniques of [2].

Suppose we have a solution to (1). Then, working 2-adically, we readily obtain that $x$ is odd, say $x = 2x_0 + 1$. It follows that

$$0 < \sqrt{2} - \frac{y}{2^{x_0}} = \frac{z^2(x^2 - 2)}{2^{x_0} (2^{x_0+1/2} + y)} < \frac{z^2(x_0^2 + x_0)}{2^{2x_0-3/2}}.$$
On the other hand, from Corollary 1.6 of Bauer and Bennett [1], either $x_0 \in \{3, 7, 8\}$, or
$$\sqrt{2} - \frac{y}{2x_0} > 2^{-1.48x_0}.$$  
In the latter case, if $z = 1$, we thus have
$$2^{0.52x_0} < 2^{1.5} (x_0^2 + x_0)$$
and so, using calculus, $x_0 \leq 19$. A quick check yields the main result of [2].

The same argument with a little more work, applying Corollary 1.6 of [1] to either
$$\sqrt{2} - \frac{y}{2x_0} \quad \text{or} \quad \sqrt{2} - \frac{z x}{2x_0}$$
implies the following:

**Theorem.** For any solution to equation (1) in positive integers $x$, $y$ and $z$, we either have $(x, y, z) \in \{(3, 1, 1), (5, 3, 1), (7, 9, 1), (9, 14, 2), (13, 3, 7)\}$ or may conclude that
$$\min \{y, z\} > 2^{x/8}.$$  

Gica and Luca conjecture that equation (1) has only the solutions $(x, y, z) = (3, 1, 1), (5, 3, 1), (7, 9, 1) \text{ and } (13, 3, 7)$ in odd integers $x, y$ and $z$ with $x^2 - 2$ prime. The above result confirms this in case either $y$ or $z$ are “small”. Indeed, an almost immediate consequence of this theorem (together with routine computation) is that (1) has only the following solutions with $z < 10^8$:

$$(x, y, z) = (3, 1, 1), (5, 3, 1), (7, 9, 1), (9, 14, 2), (11, 12, 4), (13, 3, 7), (15, 136, 8), (17, 240, 16), (21, 1324, 28), (23, 2496, 64), (25, 5568, 64), (27, 11456, 64), (31, 33792, 1024), (31, 29502, 1154), (33, 71300, 1796), (33, 63488, 2048), (37, 318048, 5152), (39, 741152, 544), (39, 376832, 16384), (45, 1934080, 124672), (45, 655360, 131072), (47, 8639880, 173048), (51, 23078252, 813316), (51, 7995392, 917504), (55, 189629256, 151672), (63, 2181038080, 33554432), (63, 230976400, 31307768), (63, 2043658240, 35667968), (65, 4490035200, 62947328), (65, 4227858432, 67108864), (65, 3949423632, 71012336), (69, 24270711104, 16066112).

**References**

