ON SETS WITH MORE RESTRICTED SUMS THAN DIFFERENCES

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Abstract

Given a finite set $A$ of integers, we define its restricted sumset $A + A$ to be the set of sums of two distinct elements of $A$ - a subset of the sumset $A + A$ - and its difference set $A - A$ to be the set of differences of two elements of $A$. We say $A$ is a restricted-sum-dominant set if $|A + A| > |A - A|$. Though intuition suggests that such sets should be rare, we present various constructions of such sets and prove that a positive proportion of subsets of $\{0, 1, \ldots, n - 1\}$ are restricted-sum-dominant sets. As a by-product, we improve on the previous record for the maximum value of $\ln(|A + A|)/\ln(|A - A|)$, and give some related discussion.

1. Introduction

Let $A$ be a finite set of integers. We define its sumset $A + A$ to be $\{a + b : a, b \in A\}$, its difference set $A - A$ to be $\{a - b : a, b \in A\}$ and its restricted sumset $A + A$ to be $\{a + b : a \neq b, a, b \in A\}$. It is a natural intuition that, since addition is commutative but subtraction is not, that ‘often’ we should have $|A + A| \leq |A - A|$. However it has been known for some time that this is not always the case: for example, the set $C = \{0, 2, 3, 4, 7, 11, 12, 14\}$, which is attributed to Conway, has $|C + C| = 26$, but $|C - C| = 25$. In this paper, sets with this property are called sum-dominant: in some other literature, they are described as MSTD (for ‘more sums than differences’) sets, see, e.g., Nathanson [6]. It is now known by work of Martin and O’Bryant [5] that sum-dominant sets are less rare than they might initially appear: they prove that, for $n \geq 15$, the proportion of subsets of $\{0, 1, 2 \ldots n - 1\}$
which are sum-dominant is at least $2 \times 10^{-7}$. The constant was sharpened, and the existence of a limit shown, by Zhao [11].

In this paper we investigate what might appear to be an even more demanding condition on a set, namely what we will call the restricted-sum-dominant property.

**Definition 1.** A set $A$ of integers is said to be *restricted-sum-dominant* if $|\hat{A} \mathbin{+} A| > |A - A|$. There are examples of this. For example, we find the set from Hegarty [3]

$$A_{15} = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25, 29, 32, 33, 37, 40, 41, 42, 44, 45\}$$

has $|\hat{A}_{15} \mathbin{+} A_{15}| = 86$ whilst $|A_{15} - A_{15}| = 83$.

Clearly any restricted-sum-dominant set is sum-dominant. The converse is false as Conway’s set is sum-dominant but not restricted-sum-dominant ($|C \mathbin{+} C| = 21$).

Note that the property of being restricted-sum-dominant is preserved when we apply a bijection of the form $x \to ax + b$ with $a, b \in \mathbb{Z}, a \neq 0$. It therefore suffices to consider sets $A \subset \mathbb{Z}$ with $\min(A) = 0$ and $\gcd(A) = 1$. We shall refer to such sets as being *normalised*.

The organisation of this paper is as follows. In Section 2 we exhibit several sequences of restricted-sum-dominant sets, addressing some natural questions about the relative sizes of the restricted sumset and difference sets. In Section 3, we show that a strictly positive proportion of subsets of $\{0, 1, 2, \ldots, n-1\}$ are restricted-sum-dominant sets. In Section 4 we obtain a new record high value of each of

$$f(A) = \frac{\ln(|A + A|)}{\ln(|A - A|)} \quad \text{and} \quad g(A) = \frac{\ln(|A \mathbin{+} A|/|A|)}{\ln(|A - A|/|A|)}$$

and give some related discussion. Finally, in Section 5 we improve somewhat the bounds on the order of the smallest restricted-sum-dominant set.

We shall, slightly unusually, use the notation $[a, b]$, when $a < b$ are integers, to denote $\{a, a + 1, \ldots, b\}$.

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2. Explicit Sequences of Restricted-Sum-Dominant Sets

Our first sequence of restricted-sum-dominant sets arose by considering the set $B = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25, 28, 30, 32, 33\}$ which appears in [7] and [9] as a set of integers with $|B \mathbin{+} B| > |(B - B) \setminus \{0\}|$. We then noted that replacing 33 with 29 gives a 16-element restricted-sum-dominant set (which will be $T_3^3$ below). To get the subsequent terms of the sequence, we used (here and elsewhere in the paper) the idea from [9], Conjecture 6, that repetition of certain so-called interior
blocks when the set is written in order as a sequence of differences can increase the size of the sumset more than the difference set: see [9] for details.

**Theorem 2.** For every integer \( j \geq 1 \) we define

\[
T_j^* = \{0, 2\} \cup \{1, 9, \ldots, 1 + 8j\} \cup \{4, 12, \ldots, 4 + 8j\} \\
\cup \{5, 13, \ldots, 5 + 8j\} \cup \{6 + 8j, 8(j + 1)\}.
\]

Then

\[
T_j^* \star T_j^* = [1, 6 + 8(2j + 1)] \setminus \{8, 8(2j + 1)\}, \\
T_j^* + T_j^* = [0, 8(2j + 2)] \setminus \{7 + 8(2j + 1)\} \text{ and} \\
T_j^* - T_j^* = [-8(j + 1), 8(j + 1)] \setminus \{\pm 6, \ldots \pm (6 + 8(j - 1))\}.
\]

**Proof:** We deal first with the restricted sumset. Since \( 0 \in T_j^* \), \( T_j^* \setminus \{0\} \subseteq T_j^* \star T_j^* \), giving all elements congruent to 1, 4 or 5 mod 8 less than \( 8(j + 1) \). Also

\[
8(j + 1) \oplus \{1, 9, \ldots, 1 + 8j\} = \{1 + 8(j + 1), \ldots, 1 + 8(2j + 1)\}
\]

\[
8(j + 1) \oplus \{4, 12, \ldots, 4 + 8j\} = \{4 + 8(j + 1), \ldots, 4 + 8(2j + 1)\}
\]

\[
8(j + 1) \oplus \{5, 13, \ldots, 5 + 8j\} = \{5 + 8(j + 1), \ldots, 5 + 8(2j + 1)\}
\]

so \( T_j^* \star T_j^* \) contains all the elements congruent modulo 8 to 1, 4 or 5 stated. For integers congruent to 2 modulo 8 the restricted sumset contains 0+2 and

\[
\{1, 9, \ldots, 1 + 8j\} \oplus \{1, 9, \ldots, 1 + 8j\} = \{10, 18, \ldots, 2 + 8(2j - 1)\}
\]

gives most of the rest: the two missing elements are \((4+8j)+(6+8j)=2+8(2j+1)\)

and \(4+8(j-1)+6+8j=2+8(2j)\).

For integers congruent to 3 modulo 8, note that

\[
\{1, 9, \ldots, 1 + 8j\} \oplus \{2\} = \{3, 11, \ldots, 3 + 8j\}
\]

and

\[
(6 + 8j) \oplus \{5, 13, \ldots, 5 + 8j\} = \{3 + 8(j + 1), \ldots, 3 + 8(2j + 1)\}.
\]

For integers congruent to 6 modulo 8,

\[
\{1, 9, \ldots, 1 + 8j\} \oplus \{5, 13, \ldots, 5 + 8j\} = \{6, 14, \ldots, 6 + 8(2j)\}
\]

and \((6+8j)+8(j+1)=6+8(2j+1)\in T_j^* \star T_j^* \) also. The elements congruent to 7 modulo 8 are obtained from

\[
(2) + \{5, 13, \ldots, 5 + 8j\} = \{7, 15, \ldots, 7 + 8j\}
\]

and

\[
(6 + 8j) + \{1, 9, \ldots, 1 + 8j\} = \{7 + 8j, \ldots, 7 + 8(2j)\}
\]
in \( T'_j + T'_j \). Finally, the required multiples of 8 are obtained from
\[
\{4, 12, \ldots, 4 + 8j\} + \{4, 12, \ldots, 4 + 8j\} = \{16, 24, \ldots, 8(2j)\}.
\]
Finally we note that the alleged omitted elements 0, 8 and \( 8(2j + 1) \) are not in \( T'_j + T'_j \). The claim for 0 is clear, the only way to get 8 is as \( 4 + 4 \) which is not a restricted sum, for \( 8(2j + 1) \) the large elements of \( T'_j \) are \( 5 + 8j, 6 + 8j, 8(j + 1) \in T'_j \) but \( 3 + 8j, 2 + 8j, 8j \notin T'_j \) so it could only be obtained as \( (4 + 8j) + (4 + 8j) \) which is not a restricted sum.

Next we address the sumset \( T'_j + T'_j \). All we need do here is note that \( 0 = 0 + 0, \)
\( 8 = 4 + 4, 7 + 8(2j + 1) \) is still not attained and that \( 8(2j + 2) = 8(j + 1) + 8(j + 1) \).

We finally deal with \( T'_j - T'_j \). Given that \( d \in T_j - T_j \iff -d \in T_j - T_j \) it suffices
to consider the positive differences. Firstly we show that \( \{6, \ldots, 6 + 8(j - 1)\} \notin T'_j - T'_j \). Given that \( T'_j \) has the form
\[
T'_j = \{0, 1 + 8x, 2, 4 + 8y, 5 + 8z, 6 + 8j, 8(j + 1)\}
\]
(where \( 0 \leq x, y, z, \leq j \)), considering the difference set \( T'_j - T'_j \) we see that the only
difference of the form \( 6 + 8t \) (where \( t \) is a non-negative integer) is \( 6 + 8j \), as stated.

To confirm \( T'_j - T'_j \) does contain the other elements in the interval specified, note
that, as \( 0 \in T'_j, T'_j \subseteq T'_j - T'_j \). The other elements are obtained as follows:
\[
\{1, 9, \ldots, 1 + 8j\} - (1) = \{0, 9, \ldots, 8j\}
\{4, 12, \ldots, 4 + 8j\} - 1 = \{3, 11, \ldots, 3 + 8j\}
\{4, 12, \ldots, 4 + 8j\} - 2 = \{2, 10, \ldots, 2 + 8j\}
\{12, 20, \ldots, 4 + 8j\} - (5) = \{7, 15, \ldots, 7 + 8(j - 1)\}
8(j + 1) - (1) = 7 + 8j.
\]
Thus all the elements of the right-hand side are in \( T'_j - T'_j \) as required. \( \Box \)

**Corollary 3.** For every integer \( j \geq 1 \) the set \( T'_j \subset \mathbb{Z} \) has
\[
| T'_j | = 3j + 7, \quad | T'_j + T'_j | = 16j + 12, \quad | T'_j + T'_j | = 16j + 16 \quad \text{and} \quad | T'_j - T'_j | = 14j + 17.
\]
Therefore
\[
| T'_j + T'_j | - | T'_j - T'_j | = 2j - 5, \quad | T'_j + T'_j | - | T'_j - T'_j | = 2j - 1
\]
and \( T'_j \) is an restricted-sum-dominant set for every integer \( j \geq 3 \).

\( T'_j \) of order 16 is one of the two smallest restricted-sum-dominant sets we have.

The set \( T'_j \) has a superset \( T_j = T'_j \cup 1 + 8(j + 1) \), which is also restricted-sum-dominant for \( j \geq 3 \):
**Theorem 4.** For every integer \( j \geq 1 \) define

\[
T_j = \{0, 2\} \cup \{1, 9, \ldots, 1 + 8(j + 1)\} \cup \{4, 12, \ldots, 4 + 8j\} \\
\cup \{5, 13, \ldots, 5 + 8j\} \cup \{6 + 8j, 8(j + 1)\}.
\]

Then

\[
T_j + T_j = [1, 1 + 8(2j + 2)] \backslash \{8, 8(2j + 1), 8(2j + 2)\},
\]

\[
T_j + T_j = [0, 2 + 8(2j + 2)] \text{ and}
\]

\[
T_j - T_j = [-1 + 8(j + 1)), 1 + 8(j + 1)] \backslash \{\pm 6, \ldots, \pm (6 + 8(j - 1))\}.
\]

**Proof.** Firstly since \( T_j \supset T_j' \) we have \( T_j + T_j \supset [1, 6 + 8(2j + 1)] \backslash \{8, 8(2j + 1)\} \). With \( 1 + 8(j + 1) \in T_j \) we now also have that

\[
8(j + 1) + (1 + 8(j + 1)) = 1 + 8(2j + 2) \quad \text{and}
\]

\[
(6 + 8j) + (1 + 8(j + 1)) = 7 + 8(2j + 1)
\]

are in \( T_j + T_j \) as well. Furthermore

\[
(1 + 8(j + 1)) + (1 + 8(j + 1)) = 2 + 8(2j + 2) \in T_j + T_j.
\]

This completes the claims for the sumset and restricted sumset, noting that clearly 8 and 8(2j + 2) are not in \( T_j + T_j \) and checking that 8(2j + 1) \( \not\in T_j + T_j \).

As regards the difference set, with \( 0 \leq x \leq j + 1 \) the positive differences resulting from the introduction of the new element have the form

\[
(1 + 8(j + 1)) - \{0, 2, 1 + 8x, 4 + 8y, 5 + 8z, 6 + 8j, 8(j + 1)\}
\]

\[
= \{1 + 8(j + 1), 8j + 7, 8(j - x + 1), 8(j - y) + 5, 8(j - z) + 4, 3, 1, 0\}.
\]

This shows that \( T_j - T_j = T_j' - T_j' \cup \{1 + 8(j + 1)\} \) and the result follows. \( \Box \)

**Corollary 5.** For every integer \( j \geq 1 \) the set \( T_j \subset \mathbb{Z} \) has

\[
|T_j| = 3j + 8, \ |T_j + T_j| = 16j + 14, \ |T_j + T_j| = 16j + 19 \quad \text{and} \quad |T_j - T_j| = 14j + 19.
\]

Therefore

\[
|T_j + T_j| - |T_j - T_j| = 2j - 5, \quad |T_j + T_j| - |T_j - T_j| = 2j
\]

and \( T_j \) is an restricted-sum-dominant set for every integer \( j \geq 3 \).

In [5], Martin and O’Bryant construct, for all integers \( x \), subsets \( S \) of \( [0, 17|x|] \) with \( |S + S| - |S - S| = x \). Corollary 3 shows that for each positive odd integer \( x \) there is \( T_j' \subset \mathbb{Z} \) with \( |T_j' + T_j'| - |T_j' - T_j'| = x \), and Corollary 5 shows each positive
even integer can be expressed as the difference of the cardinalities of the sumset and the difference set of some $T_j \subset \mathbb{Z}$.

Recall that the diameter of a finite set $A$ of integers is $\max(A) - \min(A)$. There is some interest in finding sets of integers of small diameter with prescribed relationships between the order of the sumset (or restricted sumset) and the difference set: see, e.g., [5] Theorem 4 where sets $S_x$ of diameter at most $17|x|$ are constructed with $|S_x + S_x| - |S_x - S_x|$ equal to $x$. Our sets $T_j$ and $T'_j$ have respective diameters $8j + 8$ and $8j + 9$, which is smaller than the sets $S_x$ in [5] for $j \geq 3$.

Further Corollary 5 makes it clear that the difference between the size of the restricted sumset and the difference set can be any odd positive integer. We will get any even difference for $|A + A| - |A - A|$ in our next construction. This was motivated by the sum-dominant (but not restricted-sum-dominant) set called $A_{13} = \{0, 1, 2, 4, 7, 8, 12, 14, 15, 18, 19, 20\}$ in Hegarty [3]. We exhibit, addressing his remark about the desirability of generalising $A_{13}$, two infinite sequences of (eventually) restricted-sum dominant sets derived from $A_{13}$ (which shall be our $R_1$).

**Theorem 6.** For each integer $j \geq 1$ define $R_j \subset \mathbb{Z}$ to be the set

$$R_j = \{1, 4\} \cup \{0, 12, \ldots, 12j\} \cup \{2, 14, \ldots, 2 + 12j\} \cup \{7, 19, \ldots, 7 + 12j\} \cup \{8, 20, \ldots, 8 + 12j\} \cup \{3 + 12j, 6 + 12j\}.$$  

For each integer $j \geq 2$ we have

$$R_j + R_j = [1, 3 + 12(2j + 1)] \setminus \{(17, \ldots, 5 + 12(j - 1)) \cup \{12(2j), 12(2j + 1)\}\},$$

$$R_j + R_j = [0, 4 + 12(2j + 1)] \setminus \{(17, \ldots, 5 + 12(j - 1))\} \quad \text{and}$$

$$R_j - R_j = [-(8 + 12j), 8 + 12j] \setminus \{\pm 9, \ldots, \pm (9 + 12(j - 1))\}.$$  

**Proof.** We first verify the claim for the restricted sumset. For multiples of 12,

$$\{0, 12, \ldots, 12j\} + \{0, 12, \ldots, 12j\} = \{12, 24, \ldots, 12(2j - 1)\}.$$  

The elements congruent to 1 modulo 12 are given by

$$(1) + \{0, 12, \ldots, 12j\} = \{1, 13, \ldots, 1 + 12j\}.$$  

and

$$(6 + 12j) + \{7, 19, \ldots, 7 + 12j\} = \{1 + 12(j + 1), \ldots, 1 + 12(2j + 1)\}.$$  

For those congruent to 2 modulo 12

$$\{0, 12, \ldots, 12j\} + \{2, 14, \ldots, 2 + 12j\} = \{2, 14, \ldots, 2 + 12(2j)\}.$$  

and also $(6 + 12j) + (8 + 12j) = 2 + 12(2j + 1) \in R_j + R_j$. For 3 modulo 12 clearly $3 = 1 + 2 \in R_j + R_j$ and the rest follow from

$$\{7, 19, \ldots, 7 + 12j\} + \{8, 20, \ldots, 8 + 12j\} = \{15, 27, \ldots, 3 + 12(2j + 1)\}.$$  

For elements congruent to 4 modulo 12, we clearly have that 4 and 16 are in $R_j + R_j$ as well as

$$\{8, 20, \ldots, 8 + 12j\} + \{8, 20, \ldots, 8 + 12j\} = \{28, 40, \ldots, 4 + 12(2j)\}.$$ 

The elements congruent to 6 modulo 12 in $R_j + R_j$ can be obtained as the union of

$$(4) + \{2, 14, \ldots, 2 + 12j\} = \{6, 18, \ldots, 6 + 12j\}$$

and

$$(6 + 12j) + \{0, 12, \ldots, 12j\}.$$ 

The elements congruent to 7 (respectively 8) modulo 12 are obtained from

$$\{0, 12, \ldots, 12j\} + \{7, 19, \ldots, 7 + 12j\} = \{7, 19, \ldots, 7 + 12(2j)\}.$$ 

and

$$\{0, 12, \ldots, 12j\} + \{8, 20, \ldots, 8 + 12j\} = \{8, 20, \ldots, 8 + 12(2j)\}.$$ 

For 9 (respectively 10) modulo 12 use

$$\{2, 14, \ldots, 2 + 12j\} + \{7, 19, \ldots, 7 + 12j\} = \{9, 21, \ldots, 9 + 12(2j)\}$$

respectively

$$\{2, 14, \ldots, 2 + 12j\} + \{8, 20, \ldots, 8 + 12j\} = \{10, 22, \ldots, 10 + 12(2j)\}.$$ 

Finally the elements congruent to 11 modulo 12 are obtained from

$$(4) + \{7, 19, \ldots, 7 + 12j\} = \{11, 23, \ldots, 11 + 12j\}$$

and

$$(3 + 12j) + \{8, 20, \ldots, 8 + 12j\} = \{11 + 12j, \ldots, 11 + 12(2j)\}.$$ 

To see that the restricted sumset does not contain any of $\{17, \ldots, 5 + 12(j - 1)\}$, note that none of the sumsets of the progressions with common difference 12 give elements which are congruent to 5 modulo 12 and neither can translates of the progressions by 1 or 4). The remaining elements congruent to 5 modulo 12 are obtained as clearly 5 $\in R_j + R_j$, and also

$$(3 + 12j) + \{2, 14, \ldots, 2 + 12j\} = \{5 + 12j, \ldots, 5 + 12(2j)\} \subseteq R_j + R_j.$$ 

Finally, to see that $R_j + R_j$ does not contain $12(2j)$ or $12(2j + 1)$, note that it is impossible to obtain $12(2j)$ as a sum of distinct elements of $R_j$ since the only elements of $R_j$ greater than $12j$ are $S = \{2 + 12j, 3 + 12j, 6 + 12j, 7 + 12j, 8 + 12j\}$ but none of the numbers in $2(12j) - S$ (namely $10 + 12(j - 1), 9 + 12(j - 1)$,
6 + 12(j - 1), 5 + 12(j - 1), 4 + 12(j - 1)) are in \( R_j \). Further as \( 12(j + 1) \not\in R_j \)
\( 12(2j + 1) \) is excluded from \( R_j \). This completes the argument for \( R_j \).

However, we do have that \( 12j + 12j = 12(2j) \in R_j \) and \( (6 + 12j) + (6 + 12j) = 12(2j + 1) \in R_j \). Since we readily see that none of the numbers congruent to 7 mod 12 ruled out of \( R_j \) are in \( R_j + R_j \) either, the subset is as stated.

To confirm the claim for the difference set as before we consider the positive differences. Writing \( R_j \) as

\[
\{1, 4, 12w, 2 + 12x, 7 + 12y, 8 + 12z, 3 + 12j, 6 + 12j\}
\]

the remainders which occur in \( R_j - R_j \) are exactly the set \([0, 11]\setminus\{9\}\). On the other hand, to see that \( R_j - R_j \) contains all the claimed differences, note that as \( 0 \in R_j \)
we have \( R_j \subset R_j - R_j \). Also the right hand sides of

\[
\begin{align*}
\{0, 12, \ldots, 12j\} - (1) &= \{-1, 11, \ldots, 11 + 12(j - 1)\} \\
\{2, 14, \ldots, 2 + 12j\} - (1) &= \{1, 13, \ldots, 1 + 12j\} \\
\{7, 19, \ldots, 7 + 12j\} - (4) &= \{3, 15, \ldots, 3 + 12j\} \\
\{8, 20, \ldots, 8 + 12j\} - (4) &= \{4, 16, \ldots, 4 + 12j\} \\
\{7, 19, \ldots, 7 + 12j\} - (2) &= \{5, 17, \ldots, 5 + 12j\} \\
\{7, 19, \ldots, 7 + 12j\} - (1) &= \{6, 18, \ldots, 6 + 12j\} \\
\{2, 14, \ldots, 2 + 12j\} - (4) &= \{-2, 10, \ldots, 10 + 12(j - 1)\}.
\end{align*}
\]

are in the difference set which completes the claim.

\[ \square \]

**Corollary 7.** For every integer \( j \geq 2 \) the set \( R_j \subset \mathbb{Z} \) has

\[ |R_j| = 4j + 8, \quad |R_j + R_j| = 23j + 14, \quad |R_j + R_j| = 23j + 18 \quad \text{and} \quad |R_j - R_j| = 22j + 17. \]

Therefore

\[ |R_j + R_j| - |R_j - R_j| = j - 3, \quad |R_j + R_j| - |R_j - R_j| = j + 1 \]

and \( R_j \) is an restricted-sum-dominant set for every integer \( j \geq 4 \).

This indeed confirms that any positive integer can be obtained as \( |R_j + R_j| - |R_j - R_j| \).

Our fourth sequence of sets, the \( M_j \)'s, also has \( R_1 \) (Hegarty’s \( A_{13} \)) as its first member, but this time we focus not on prescribing \( |M_j + M_j| - |M_j - M_j| \) but instead on getting a reduced diameter \( 9 + 11j \) rather than the diameter \( 8 + 12j \) of \( R_j \). (We were first led to this family by considering Marica’s sum-dominant set \([4] M = \{1, 2, 3, 5, 8, 9, 13, 15, 16\} \), normalising it and trying to expand it to a restricted-sum-dominant set).
Theorem 8. For $j \geq 1$ define

$$M_j = \{0, 2\} \cup \{1, 12, \ldots, 1 + 11j\} \cup \{4, 15, \ldots, 4 + 11j\}$$

$$\cup \{7, 18, \ldots, 7 + 11j\} \cup \{8, 19, \ldots, 8 + 11j\} \cup \{3 + 11j, 9 + 11j\}$$

We then have that

$$M_j + M_j = [1, 6 + 11(2j + 1)] \setminus \{3 + 11(2j + 1)\}$$

$$M_j + M_j = [0, 7 + 11(2j + 1)] \text{ and}$$

$$M_j - M_j = [- (9 + 11j), 9 + 11j] \setminus \{ \pm 9, \ldots, \pm (9 + 11(j - 1))\}.$$
For the case $a = 3$

$$\{7,18,\ldots,7+11j\} \oplus \{7,18,\ldots,7+11j\} = \{25,36,\ldots,3+11(2j)\}$$

and $3 = 1+2, 14 = 2+12$ are in $M_j \hat{\oplus} M_j$.

For the case $a = 7$

$$\{0\} + \{7,18,\ldots,7+11j\} = \{7,18,\ldots,7+11j\}$$
$$\{3+11j\} + \{4,15,\ldots,4+11j\} = \{7+11j,\ldots,7+11(2j)\}.$$

For the case $a = 8$

$$\{1,12,\ldots,1+11j\} \oplus \{7,18,\ldots,7+11j\} = \{8,19,\ldots,8+11(2j)\}.$$ 

For the case $a = 9$

$$\{1,12,\ldots,1+11j\} \oplus \{8,19,\ldots,8+11j\} = \{9,20,\ldots,9+11(2j)\}.$$ 

For $a = 10$

$$\{2\} \oplus \{8,19,\ldots,8+11j\} = \{10,21,\ldots,10+11j\}$$
$$\{3+11j\} \oplus \{7,18,\ldots,7+11j\} = \{10+11j,\ldots,10+11(2j)\}.$$ 

For $a = 11$

$$\{4,15,\ldots,4+11j\} \oplus \{7,18,\ldots,7+11j\} = \{11,22,\ldots,11+11(2j)\}.$$ 

To see that $3 + 11(2j + 1) \notin M_j \hat{\oplus} M_j$, if it did not we would have a sum of the form $(a + 11j) + (c + 11j) = 14 + 22j$ from elements of $M_j$ with $a + c = 14$, however, since $a$ and $c$ are distinct elements of $\{1,3,4,7,8,9\}$ this is impossible and hence $3 + 11(2j + 1) \notin M_j \hat{\oplus} M_j$. This confirms the claim for the restricted sunset. Furthermore for each $m \in M_j$ the sumset contains $0, 2(7 + 11j) = 3 + 11(2j + 1)$ and $2(9 + 11j) = 7 + 11(2j + 1)$ which completes the claim for the sunset.

For the difference set to see that $\{\pm 9,\ldots,\pm(9+11(j-1))\} \notin M_j - M_j$ let

$$M_j = \{0,2,1+11w,4+11x,7+11y,8+11z,3+11j,9+11j\},$$

where $0 \leq w, x, y, z \leq j$. It suffices to consider just the positive differences. Calculation of $M_j - M_j$ reveals that the only positive difference congruent to 9 modulo 11 is $(9+11j) - 0$, which is outside the range claimed.

To see that $M_j - M_j$ contains the remaining elements in the interval, firstly note that as $0 \in M_j$ we have $M_j - M_j \supset M_j$. Furthermore $M_j - M_j$ also contains the
right-hand sides of the following:

\[
\begin{align*}
\{1, 12, \ldots, 1 + 11j\} \setminus (1) &= \{0, 11, \ldots, 11j\} \\
\{4, 15, \ldots, 4 + 11j\} \setminus (1) &= \{3, 14, \ldots, 3 + 11j\} \\
\{7, 18, \ldots, 7 + 11j\} \setminus (1) &= \{6, 17, \ldots, 6 + 11j\} \\
\{1, 12, \ldots, 1 + 11j\} \setminus (2) &= \{-1, 10, 21, \ldots, 10 + 11(j - 1)\} \\
\{4, 15, \ldots, 4 + 11j\} \setminus (2) &= \{2, 13, \ldots, 2 + 11j\} \\
\{7, 18, \ldots, 7 + 11j\} \setminus (2) &= \{5, 16, \ldots, 5 + 11j\} \\
9 + 11j - 0 &= 9 + 11j.
\end{align*}
\]

This completes the claim of the theorem. \(\square\)

**Corollary 9.** For every integer \(j \geq 1\) the set \(M_j \subset \mathbb{Z}\) has

\[
|M_j| = 4j + 8, \quad |M_j + M_j| = 22j + 16, \quad |M_j + M_j| = 22j + 19 \quad \text{and} \quad |M_j - M_j| = 20j + 19.
\]

Hence

\[
|M_j + M_j| - |M_j - M_j| = 2j - 3, \quad |M_j + M_j| - |M_j - M_j| = 2j
\]

and \(M_j\) is an restricted-sum-dominant set for every \(j \geq 2\).

Note that the set \(M_2\) has slightly smaller diameter 31 than the other 16-element restricted-sum-dominant set \(T_3^{16}\).

Martin and O’Bryant refer to sets with \(|A + A| = |A - A|\) as sum-difference balanced. Similarly we can consider sets with \(|A + A| = |A - A|\) as restricted-sum-difference balanced. The results above show such sets exist (e.g., \(R_3\)). The smallest such set we have found has order 14: it is

\[
M' = \{0, 1, 2, 4, 7, 8, 12, 14, 15, 19, 22, 25, 26, 27\},
\]

so \(|M' + M'| = |[1, 53] \setminus \{43, 50\}| = 51\) and \(|M' - M'| = |[-27, 27] \setminus \{\pm 9, \pm 16\}| = 51\). We show that by taking the union of translates of \(M'\) by non-negative integer multiples of its maximum element one can obtain arbitrarily large restricted-sum-difference balanced sets.

**Lemma 10.** Let \(k \geq 2\) and \(A_0 = A = \{0 = a_1 < a_2 < \cdots < a_k = m\} \subset \mathbb{Z}\) and \(A_i = A \cup (A + m) \cup \cdots \cup (A + im)\). Then

\[
|A_i + A_i| - |A_{i-1} + A_{i-1}| = c_1 \quad \forall i \geq 2,
\]

\[
|A_i + A_i| - |A_{i-1} + A_{i-1}| = c_1 \quad \forall i \geq 1
\]

and

\[
|A_i - A_i| - |A_{i-1} - A_{i-1}| = c_2 \quad \forall i \geq 1.
\]

where \(c_1\) and \(c_2\) are positive constants.
Proof. We first note

$$|A_i \uparrow A_i| - |A_{i-1} \uparrow A_{i-1}| = |(A_i \uparrow A_i) \setminus (A_{i-1} \uparrow A_{i-1})|$$

and show that the right-hand side is a constant by showing that the set of new elements introduced on each iteration is a translate of the set of new elements introduced on the previous iteration. We have

$$A_i \uparrow A_i = \bigcup_{r,s=0}^r ((A + rm) \uparrow (A + sm)).$$

If $|r - s| \geq 2$, it is clear that $A + rm$ and $A + sm$ are disjoint so their restricted sum is just their sum. If $i - 1 \geq r = s \geq 1$, then $(A + rm) \uparrow (A + rm) = (A + (r - 1)m) + (A + (r + 1)m)$. The only case needing a little thought is $|r - s| = 1$: without loss of generality, $r = s + 1$. Then

$$(A + (s + 1)m) \uparrow (A + sm) = \{a + b + (2s + 1)m : a + m \neq b\}$$

the only way we can have $a + m = b$ is if $a = 0, b = m$, but in this case

$$(0 + (s + 1)m) + (m + sm) = (m + (s + 1)m) \uparrow (0 + sm)$$

We deduce that, for all $i \geq 2$

$$A_i \uparrow A_i = (A \uparrow A) \cup (A + (A + m)) \cup \cdots \cup (A + A + (2i - 1)m) \cup (A \uparrow A + 2im).$$

Similarly

$$A_{i-1} \uparrow A_{i-1} = (A \uparrow A) \cup (A + A + m) \cup \cdots \cup (A \uparrow A + (2i - 2)m).$$

Now some elements of $(A + A + (2i - 2)m) \setminus (A \uparrow A + (2i - 2)m)$ may be in $A + A + (2i - 3)m$ and thus in $A_{i-1} \uparrow A_{i-1}$. (Translates of $A \uparrow A$ by less than $(2i - 3)m$ need not be considered). We have

$$(A_i \uparrow A_i) \setminus (A_{i-1} \uparrow A_{i-1}) = ((A + A + (2i - 2)m) \cup (A + A + (2i - 1)m) \cup (A \uparrow A + 2im)) \setminus ((A + A + (2i - 3)m) \cup (A \uparrow A + (2i - 2)m)).$$  (1)

Likewise

$$(A_{i+1} \uparrow A_{i+1}) \setminus (A_{i+1} \uparrow A_{i}) = ((A + A + 2im) \cup (A + A + (2i + 1)m) \cup (A \uparrow A + (2i + 2)m)) \setminus ((A + A + (2i + 1)m) \cup (A \uparrow A + (2i)m)).$$  (2)

The right-hand side of (2) is a translation of the right-hand side of (1) by $2m$. (To see this, note it is easy to check for sets of integers that if $C_i + 2m = C_{i+1}$ and $D_i + 2m = D_{i+1}$, then $(C_i \setminus D_i) + 2m = (C_{i+1} \setminus D_{i+1})$: apply this with the obvious choices of $C_i$ and $D_i$). Thus

$$(A_{i+1} \uparrow A_{i+1}) \setminus (A_{i+1} \uparrow A_{i}) = ((A_i \uparrow A_i) \setminus (A_{i-1} \uparrow A_{i-1})) + 2m.$$
Since translation by a constant leaves the cardinality of the set difference unaltered it follows that
\[ |(A_{i+1 } + A_{i+1 } ) \setminus (A_i + A_i )| = |(A_i + A_i ) \setminus (A_{i+1 } + A_{i+1 } )| \]
as required.

To see that
\[ |A_i + A_i | - |A_{i-1 } + A_{i-1 }| = |A_i + A_i | - |A_{i-1 } + A_{i-1 }| \]
for all \( i \geq 1 \) we show that the number of additional elements \( A_i + A_i \) contains is constant. All the elements of
\[ (A + A ) \setminus (A + A ) \]
except for \( 2m \), which is in \( A_i + A_i \) for \( i \geq 1 \) due to \( 0+2m \), are excluded from \( A_i + A_i \) for all \( i \geq 1 \). Similarly the elements of
\[ ((A + A ) \setminus (A + A ) + 2im \]
extcept for \( 2im \) are excluded from \( A_i + A_i \). This means that for all \( i \geq 1 \)
\[ |A_i + A_i | - |A_i + A_i | = 2((A + A ) \setminus (A + A ) | - 1). \]

In other words the difference between the cardinalities of the sumset and the restricted sumset is a constant for all \( i \geq 1 \) and (3) holds.

To verify the claim for the difference set, write
\[ A_i - A_i = \cup_{j=-i}^{i-1}(A + jm). \]
Thus we have
\[ (A_i - A_i ) \setminus (A_{i-1 } - A_{i-1 } ) = (A - A - im) \cup (A - A + im) \setminus \bigcup_{j=-(i-1)}^{i-1}(A - A - jm). \]
But the only sets in \( \cup_{j=-1}^{i-1}(A - A - jm) \) which could intersect \( (A - A - im) \) or \( (A - A + im) \) are for \( j = (i-1), \ j = (i-2) \) (which will intersect \( A - A - im \) in precisely the one element \( (1-i)m), \ j = -(i-2) \) (which will intersect it in precisely the one element \( (i-1)m \) and \( j = -(i-1) \). Thus for all \( i \geq 1 \)
\[ (A_i - A_i ) \setminus (A_{i-1 } - A_{i-1 } ) = (A - (A + im)) \setminus (A - (A + (i-1)m))) \]
\[ \cup (A - A + im \setminus (A - A + (i-1)m))). \]
Similarly
\[ (A_{i+1 } - A_{i+1}) \setminus (A_i - A_i) = (A - (A + (i+m))) \setminus (A - (A + im))) \]
\[ \cup (A - A + (i+1)m \setminus (A - A + im))). \]
The sets \((A - (A + (i+1)m)) \setminus (A - (A + im))\) and \((A - A + (i+1)m) \setminus (A - A + im)\) are disjoint for all \(i \geq 1\). Also \((A - (A + (i+1)m)) \setminus (A - (A + im))\) is a translation of \((A - (A + im)) \setminus (A - (A + (i-1)m))\) by \(-m\) and \((A - A + (i+1)m) \setminus (A - A + im)\) is a translation of \((A - A + im) \setminus (A - A + (i-1)m)\) by \(m\). These translations leave the cardinalities of the sets unchanged, therefore
\[
|(A_{i+1} - A_{i+1}) \setminus (A_i - A_i)| = |(A_i - A_i) \setminus (A_{i-1} - A_{i-1})|
\]
and the overall result follows.

Setting \(M'_1 = M' \cup (M' + 27)\) we easily check
\[
|M'_1 + M'_1| = |[1, 107] \setminus \{97, 104\}| = |[-54, 54] \setminus \{\pm 36, \pm 43\}| = |M'_1 - M'_1|
\]
and \(M'_2 = M' \cup (M' + 27) \cup (M' + 54)\) gives
\[
|M'_2 + M'_2| = |[1, 161] \setminus \{151, 158\}| = |[-81, 81] \setminus \{\pm 63, \pm 70\}| = |M'_2 - M'_2|.
\]
It follows from Lemma 10 that

**Corollary 11.** There exist arbitrarily large restricted-sum-difference balanced subsets of \(\mathbb{Z}\).

Our final sequence of restricted-sum-dominant sets is constructed with a view to obtaining high values of \(f(A)\) as defined in the introduction. Again, this set is a modification of one in [9], which describes \(Q_j \setminus \{1 + 4(4j + 7)\}\) for \(j = 1, 2, 3\) as sets giving large sunset relative to the difference set. Including \(1 + 4(4j + 7)\) increases the sunset but does not change the difference set.

**Theorem 12.** Let
\[
Q_j = \{0, 2, 4, 12\} \cup \{1, 5, \ldots, 1 + 4(4j + 8)\} \cup \{24, 40, \ldots, 8 + 16j\}
\]
\[
\cup \{4 + 16(j + 1), 12 + 16(j + 1), 14 + 16(j + 1), 16(j + 2)\}
\]
for an integer \(j \geq 1\). Then
\[
Q_j + Q_j = [1, 1 + 4(8j + 16)]
\]
\[
\setminus \{8, 20, 32, 48(8j + 4), 4(8j + 8), 4(8j + 11), 4(8j + 14), 4(8j + 16)\}
\]
for \(j \geq 2\), whilst
\[
Q_j + Q_j = [0, 2 + 4(8j + 16)] \setminus \{20, 32, 4(8j + 8), 4(8j + 11)\}
\]
for \(j \geq 1\) and
\[
Q_j - Q_j = [-1 + 4(4j + 8)], 1 + 4(4j + 8)] \setminus \{6\}, \{14, \ldots, 14 + 16j\}, \{18, \ldots, 2 + 16j\}, \{26, \ldots, 10 + 16j\}, 6 + 16(j + 1)
\]
for \(j \geq 1\).
Proof. To verify these claims, consider elements of \( Q_j \) in terms of the union of
\[
Q_{\text{odd}} = \{1, 5, \ldots, 1 + 4(4j + 8)\}
\]
and
\[
Q_{\text{even}} = \{0, 2, 4, 12\} \cup \{24, \ldots, 8 + 16j\} \\
\setminus \{4 + 16(j + 1), 12 + 16(j + 1), 14 + 16(j + 1), 16(j + 2)\}.
\]
Firstly \( Q_j \cup Q_j \) contains all the odd numbers in the interval since we have
\[
(0) \uplus \{1, 5, \ldots, 1 + 4(4j + 8)\} = \{1, 5, \ldots, 1 + 4(4j + 8)\}
\]
\[
16(j + 2) \uplus \{1, 5, \ldots, 1 + 4(4j + 8)\} = \{1 + 4(4j + 8), 5 + 4(4j + 8), \\
\ldots, 1 + 4(8j + 16)\}
\]
\[
(2) \uplus \{1, 5, \ldots, 1 + 4(4j + 8)\} = \{3, 7, \ldots, 3 + 4(4j + 8)\}
\]
\[
14 + 16(j + 1) \uplus \{1, 5, \ldots, 1 + 4(4j + 8)\} = \{3 + 4(4j + 7), 7 + 4(4j + 7), \\
\ldots, 3 + 4(8j + 15)\}.
\]
The union of the right hand sides of the above is indeed
\[
\{1, 3, \ldots, 3 + 4(8j + 15), 1 + 4(8j + 16)\} = \{1, 3, \ldots, 1 + 2(4(4j + 8))\}.
\]
To see that the sumset contains all the even elements claimed, note first that
\( Q_{\text{odd}} \cup Q_{\text{odd}} \) gives the following elements congruent to 2 mod 4:
\[
Q_{\text{odd}} \cup Q_{\text{odd}} = \{6, 10, \ldots, 2 + 4(8j + 15)\} \subseteq Q_j \cup Q_j.
\]
Clearly 0 + 2 is also in \( Q_j \cup Q_j \), however whilst \( \max(Q_j + Q_j) = 2 + 4(8j + 16) \) this is not in the restricted sumset. As regards the multiples of four, clearly none of these can be obtained from \( Q_{\text{odd}} \cup Q_{\text{odd}} \) or \( Q_{\text{odd}} \cup Q_{\text{even}} \). To confirm the elements we claim to be excluded cannot be present note that \( Q_{\text{even}} \) is symmetric w.r.t. 16(j+2):
\[
Q_{\text{even}} = 16(j + 2) - Q_{\text{even}}. \text{ Hence } Q_{\text{even}} \uplus Q_{\text{even}} = 16(2j + 4) - (Q_{\text{even}} \uplus Q_{\text{even}}) \text{ and } \\
Q_{\text{even}} + Q_{\text{even}} = 16(2j + 4) - (Q_{\text{even}} + Q_{\text{even}}). \text{ The restricted sumset of the elements of } Q_{\text{even}} \text{ less than or equal to 32 is}
\]
\[
\{0, 2, 4, 12, 24\} \cup \{0, 2, 4, 12, 24\} = \{2, 4, 6, 12, 14, 16, 24, 26, 28, 36\}.
\]
Thus 0, 8, 20, 32 and 48 are excluded from \( Q_j \uplus Q_j \). Whilst \( Q_j + Q_j \) contains 0, 8 and 48 as the doubles of 0, 4 and 24 respectively, it is easy to check that neither 20 nor 32 are in \( Q_j + Q_j \). By symmetry
\[
16(2j + 4) - \{0, 8, 20, 32, 48\} = \{4(8j + 4), 4(8j + 8), 4(8j + 11), 4(8j + 14), 4(8j + 16)\}.
\]
which has empty intersection with \( Q_j \uplus Q_j \).
It remains to show that all other (relevant) multiples of 4 are in the (restricted) sumset; we consider the cases 0, 4, 8 and 12 modulo 16 separately. We have the following multiples of 16 in $Q_j + Q_j$:

$$\{24, 40, \ldots, 16j + 8\} + \{24, 40, \ldots, 16j + 8\} = \{64, 80, \ldots, 16(2j)\}$$

$$\quad (4 + 16(j + 1)) + (12 + 16(j + 1)) = 4(8j + 12) = 16(2j + 3).$$

Furthermore $Q_j + Q_j$ contains 48 and 16(2j + 1) = 2(16j + 8) and also 16(j + 2) + 16(j + 2) = 4(8j + 16) = 16(2j + 4). We already saw 16(2j + 2) = 4(8j + 8) is not in $Q_j + Q_j$.

We obtain those congruent to 4 modulo 16 from

$$\quad (12) + \{24, 40, \ldots, 16j + 8\} = \{36, 52, \ldots, 4 + 16(j + 1)\}$$

$$\quad (4) + (16(j + 2)) = 4 + 16(j + 2)$$

$$\quad (12 + 16(j + 1)) + \{24, \ldots, 8 + 16j\} = \{4 + 16(j + 3), \ldots, 4 + 16(2j + 2)\}$$

$$\quad (4 + 16(j + 1)) + (12 + 16(j + 2)) = 4 + 16(2j + 3).$$

The elements congruent to 8 modulo 16 are given by

$$\quad (0) + \{24, 40, \ldots, 8 + 16j\} = \{24, 40, \ldots, 8 + 16j\}$$

$$\quad (4) + (4 + 16(j + 1)) = 8 + 16(j + 1)$$

$$\quad (12) + (12 + 16(j + 1)) = 8 + 16(j + 2)$$

$$\quad (16(j + 2)) + \{24, 40, \ldots, 8 + 16j\} = \{8 + 16(j + 3), \ldots, 8 + 16(2j + 2)\}.$$ 

Also $(12 + 16(j + 1)) + (12 + 16(j + 1)) = 8 + 16(2j + 3) \in Q_j + Q_j$. Finally the elements congruent to 12 modulo 16 follow from

$$\quad (4) + \{24, \ldots, 8 + 16j\} = \{28, \ldots, 12 + 16j\}$$

$$\quad (0) + (12 + 16(j + 1)) = 12 + 16(j + 1)$$

$$\quad (4 + 16(j + 1)) + \{24, \ldots, 8 + 16j\} = \{12 + 16(j + 2), \ldots, 12 + 16(2j + 1)\}$$

$$\quad (12 + 16(j + 1)) + (16(j + 2)) = 12 + 16(2j + 3).$$

We now deal with the difference set. Again, it suffices to consider the non-negative differences. Since all the differences which we claim are excluded are even we need only consider differences of pairs of elements of $Q_j$ of the same parity and therefore divide into cases accordingly. The non-negative elements of $Q_{odd} - Q_{odd}$ are

$$\{0, 4, \ldots, 4(4j + 8)\}.$$ 

The even elements of $Q_j$ have the form

$$Q_{even} = \{0, 2, 4, 12, 8 + 16x, 4 + 16(j + 1), 12 + 16(j + 1), 14 + 16(j + 1), 16(j + 2)\}$$
where \( x \in \mathbb{Z} \) with \( 1 \leq x \leq j \). The positive differences of the elements of \( Q_{\text{even}} \) are

\[
\{2, 4, 8, 10, 12, 12 + 16(x - 1), 4 + 16x, 6 + 16x, 8 + 16x, \\
12 + 16(j - x), 4 + 16(j - x + 1), 6 + 16(j - x + 1), 8 + 16(j - x + 1), \\
8 + 16j, 16(j + 1), 2 + 16(j + 1), 4 + 16(j + 1), 8 + 16(j + 1), \\
10 + 16(j + 1), 12 + 16(j + 1), 14 + 16(j + 1), 16(j + 2)\}.
\]

Thus none of the differences in \( Q_j - Q_j \) have the form which we claim is excluded. To confirm the presence of the remaining differences we have that all the differences congruent to 1 modulo 4 are present since

\[
\{1, 5, \ldots, 1 + 4(4j + 8)\} - \{0\} = \{1, 5, \ldots, 1 + 4(4j + 8)\} \subseteq Q_j - Q_j.
\]

The elements congruent to 3 modulo 4 follow from

\[
\{1, 5, \ldots, 1 + 4(4j + 8)\} - \{2\} = \{-1, 3, \ldots, 3 + 4(4j + 7)\} \subseteq Q_j - Q_j.
\]

The multiples of 4 are obtained from

\[
\{1, 5, \ldots, 1 + 4(4j + 8)\} - \{1\} = \{0, 4, \ldots, 4(4j + 8)\}.
\]

For elements congruent to 2 mod 4, the only elements congruent to 2 mod 16 we are claiming to get are 2 and \( 2 + 16(j + 1) \); 2 is clearly in, and \( 2 + 16(j + 1) = 14 + 16(j + 1) - 12 \).

The elements congruent to 6 modulo 16 can be obtained from

\[
\{24, 40, \ldots, 8 + 16j\} - \{2\} = \{22, 38, \ldots, 6 + 16j\}.
\]

The only elements congruent to 10 mod 16 we are claiming are \( 10 + 16(j + 1) = 12 + 16(j + 1) - 2 \) and \( 10 = 12 - 2 \). Finally, the only element congruent to 14 mod 16 we claim is present is \( 14 + 16(j + 1) \in Q_j \).

**Corollary 13.** For the set \( Q_j \) defined above we have

\[
|Q_j| = 5j + 17, |Q_j + Q_j| = 32j + 56 \text{ for } j \geq 2, |Q_j + Q_j| = 32j + 63 \text{ for } j \geq 1,
\]

\[
|Q_j - Q_j| = 26j + 61 \text{ for } j \geq 1
\]

(and \( |Q_1 + Q_1| = 90 \)). Thus \( Q_j \) is an restricted-sum-dominant set for all \( j \geq 1 \).

**3. The Proportion of Restricted-Sum-Dominant Sets Is Strictly Positive**

Martin and O’Bryant prove that for \( n \geq 15 \) the number of sum-dominant subsets of \([0, n - 1]\) is at least \( (2 \times 10^{-7})2^n \) (see Theorem 1 of [5]). Their result has been
improved by Zhao [11] who shows that the proportion of sum-dominant sets tends
to a limit and that that limit is at least \(4.28 \times 10^{-4}\). In this section we will show that
the proportion of subsets of \(\{0, 1, 2, \ldots, n - 1\}\) which are restricted-sum-dominant is
bounded below by a much weaker constant. It may well be that Zhao’s techniques,
or others, can be modified to improve the result but at least a substantial piece
of computation would appear to be required and our concern at present is simply
to show that a positive proportion of sets are restricted-sum-dominant sets. Note
that the fact that a positive proportion of sets have more differences than restricted
sums is an immediate consequence of Theorem 14 in [5]. Many lemmas etc. in what
follows are very slight modifications of corresponding results in [5] and we merely
present these proofs without further comment. However the construction of the two
‘fringe sets’ \(U\) and \(L\) is notably more involved.

**Lemma 14.** Let \(n, \ell\) and \(u\) be integers such that \(n \geq \ell + u\). Fix \(L \subseteq [0, \ell - 1]\) and
\(U \subseteq [n - u, n - 1]\). Suppose \(R\) is a uniformly randomly selected subset of \([\ell, n - u - 1]\)
(where each element is chosen with probability \(1/2\)) and set \(A = L \cup R \cup U\). Then
for every integer \(k\) satisfying \(2\ell - 1 \leq k \leq n - u - 1\), we have
\[
\mathbb{P}(k \notin A^\dagger A) = \begin{cases} 
\left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{(k+1)/2-\ell}, & \text{if } k \text{ is odd,} \\
\left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{k/2-\ell}, & \text{if } k \text{ is even.}
\end{cases}
\]

**Proof.** Define an indicator variable
\[
X_j = \begin{cases} 
1, & \text{if } j \in A, \\
0, & \text{otherwise.}
\end{cases}
\]
Since \(A = L \cup R \cup U\) the \(X_j\) are independent random variables for \(\ell \leq j \leq n - u - 1\),
each taking values 0 or 1 equiprobably. For \(0 \leq j \leq \ell - 1\) and \(n - u \leq j \leq n - 1\)
the values of \(X_j\) are dictated by the choices of \(L\) and \(U\).

Now, \(k \notin A^\dagger A\) if and only if \(X_jX_{k-j} = 0\) for all \(0 \leq j \leq k/2 - 1\). \((j = k/2\)
would not give a restricted sum). The random variables \(X_jX_{k-j}\) for \(0 \leq j \leq k/2\)
are independent of each other. Hence
\[
\mathbb{P}(k \notin A^\dagger A) = \prod_{0 \leq j \leq k/2 - 1} \mathbb{P}(X_jX_{k-j} = 0).
\]
When \(k\) is odd we have
\[
\mathbb{P}(k \notin A^\dagger A) = \prod_{j=0}^{\ell-1} \mathbb{P}(X_jX_{k-j} = 0) \prod_{j=\ell}^{(k-1)/2} \mathbb{P}(X_jX_{k-j} = 0)
= \prod_{j \in L} \mathbb{P}(X_{k-j} = 0) \prod_{j=\ell}^{(k-1)/2} \mathbb{P}(X_j = 0 \text{ or } X_k-j = 0)
= \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{(k+1)/2-\ell}.
\]
When \( k \) is even

\[
\mathbb{P}(k \notin A^\hat{\omega}) = \prod_{j=0}^{\ell-1} \mathbb{P}(X_j X_{k-j} = 0) \prod_{j=\ell}^{k/2-1} \mathbb{P}(X_j X_{k-j} = 0) \\
= \prod_{j \in L} \mathbb{P}(X_{k-j} = 0) \prod_{j=\ell}^{k/2-1} \mathbb{P}(X_j = 0 \text{ or } X_{k-j} = 0) = \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{k/2-\ell}.
\]

\( \square \)

**Lemma 15.** Let \( n, \ell, u, L, U, R \) and \( A \) be defined as in Lemma 14. Then for every integer \( k \) satisfying \( n + \ell - 1 \leq k \leq 2n - 2u - 1 \), we have

\[
\mathbb{P}(k \notin A^\hat{\omega}) = \begin{cases} 
\left(\frac{1}{4}\right)^{|U|} \left(\frac{3}{4}\right)^{n-(k+1)/2-u}, & \text{if } k \text{ is odd}, \\
\left(\frac{1}{4}\right)^{|U|} \left(\frac{3}{4}\right)^{n-1-k/2-u}, & \text{if } k \text{ is even}.
\end{cases}
\]

**Proof.** This is similar to the previous lemma, but we consider different intervals for the summands. For \( k \) odd, we have

\[
\mathbb{P}(k \notin A^\hat{\omega}) = \prod_{j=(k+1)/2}^{n-u-1} \mathbb{P}(X_j X_{k-j} = 0) \prod_{j=n-u}^{n-1} \mathbb{P}(X_j X_{k-j} = 0) \\
= \prod_{j=(k+1)/2}^{n-u-1} \mathbb{P}(X_j = 0 \text{ or } X_{k-j} = 0) \prod_{j \in U} \mathbb{P}(X_{k-j} = 0) \\
= \left(\frac{3}{4}\right)^{n-(k+1)/2-u} \left(\frac{1}{2}\right)^{|U|}.
\]

For \( k \) even, as \( k = k/2 + k/2 \) is forbidden,

\[
\mathbb{P}(k \notin A^\hat{\omega}) = \prod_{j=k/2+1}^{n-u-1} \mathbb{P}(X_j X_{k-j} = 0) \prod_{j=n-u}^{n-1} \mathbb{P}(X_j X_{k-j} = 0) \\
= \prod_{j=k/2+1}^{n-u-1} \mathbb{P}(X_j = 0 \text{ or } X_{k-j} = 0) \prod_{j \in U} \mathbb{P}(X_{k-j} = 0) \\
= \left(\frac{3}{4}\right)^{n-1-k/2-u} \left(\frac{1}{2}\right)^{|U|}.
\]

\( \square \)

**Proposition 16.** Let \( n, \ell \) and \( u \) be integers such that \( n \geq \ell + u \). Fix \( L \subseteq [0, \ell - 1] \) and \( U \subseteq [n - u, n - 1] \). Suppose \( R \) is a uniformly randomly selected subset of \([\ell, n - u - 1]\) (where each element is chosen, independently of all other elements,
with probability 1/2) and set $A = L \cup R \cup U$. Then for every integer $k$ satisfying $2\ell - 1 \leq n - u - 1$,

$$P([2\ell - 1, n - u - 1] \cup [n + \ell - 1, 2n - 2u - 1] \subseteq A^\perp A) > 1 - 8(2^{-|L|} + 2^{-|U|}).$$

**Proof.** We crudely estimate

$$P([2\ell - 1, n - u - 1] \cup [n + \ell - 1, 2n - 2u - 1] \not\subseteq A^\perp A)
\leq \sum_{k=2\ell-1}^{n-u-1} P(k \notin A^\perp A) + \sum_{k=n+\ell-1}^{2n-2u-1} P(k \notin A^\perp A).$$

The left summation of the line above can be bounded using Lemma 14:

$$\sum_{k=2\ell-1}^{n-u-1} P(k \notin A^\perp A) < \sum_{k \geq 2\ell-1 \atop k \text{ odd}} \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{(k+1)/2 - \ell} + \sum_{k \geq 2\ell-1 \atop k \text{ even}} \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{k/2 - \ell}
= \left(\frac{1}{2}\right)^{|L|} \sum_{m=0}^{\infty} \left(\frac{3}{4}\right)^m + \left(\frac{1}{2}\right)^{|L|} \sum_{m=0}^{\infty} \left(\frac{3}{4}\right)^m
= 8 \left(\frac{1}{2}\right)^{|L|}.$$

The summation on the right can be bounded similarly, using Lemma 15, to give

$$\sum_{k=n+\ell-1}^{2n-2u-1} P(k \notin A^\perp A) < 8 \left(\frac{1}{2}\right)^{|U|}.$$

Thus $P([2\ell, n - u - 1] \cup [n + \ell - 1, 2n - 2u - 1] \subseteq A^\perp A)$ is bounded above by $8((1/2)^{|L|} + (1/2)^{|U|})$, which is equivalent to the claim of Proposition 16.

We now come to the main result. Whilst the respective lower and upper fringes $U = \{0, 2, 3, 7, 8, 9, 10\}$ and $L = \{n - 11, n - 10, n - 9, n - 8, n - 6, n - 3, n - 2, n - 1\}$ used by Martin and O’Bryant are sufficient for the sum-dominant case these fall some way short of what is required for a restricted-sum-dominant result. However we can again use Spohn’s idea of repeating interior blocks. After a few iterations we get the new fringes, which we shall henceforth refer to as $L$ and $U$, to fit with the earlier lemmas. Thus from now on

$$L = \{0, 2, 3, 7, 9, 10, 14, 16, 17, 21, 23, 24, 28, 30, 31, 35, 37, 38, 42, 44, 45, 49, 51, 52, 56, 57, 58, 59, 60\},$$

$$U = n - \{59, 58, 57, 55, 52, 51, 50, 48, 45, 44, 43, 41, 38, 37, 36, 34, 31, 30, 29, 27, 24, 23, 22, 20, 17, 16, 15, 13, 10, 9, 8, 6, 3, 2, 1\}.$$

**Theorem 17.** For $n \geq 120$, the number of restricted-sum-dominant subsets of $[0, n - 1]$ is at least $(7.52 \times 10^{-37})2^n$. 
Proof. With $L$ and $U$ as just defined, one can check that
\[
U - L = [n - 119, n - 1] \setminus \{n - 7, n - 14, n - 21, n - 28, n - 35, n - 42, n - 49, n - 56\}.
\]
Now since $n - 7, n - 14, n - 21, n - 28, n - 35, n - 42, n - 49, n - 56 \notin U - L$ it follows that
\[
\pm(n - 7), \pm(n - 14), \pm(n - 21), \pm(n - 28), \pm(n - 35), \pm(n - 42), \pm(n - 49), \pm(n - 56) \notin A - A \subseteq \{-n, -1, 1, n\}. 
\]
With eight pairs of differences excluded from $A - A$ we have $|A - A| \leq 2n - 17$. On the other hand one can check
\[
L \uparrow L = [0, 120] \setminus \{0, 1, 4, 6, 8, 15, 22, 29, 36, 43, 50, 120\}
\]
\[
U \uparrow L = U + L = [n - 59, n + 59]
\]
\[
U \uparrow U = [2n - 118, 2n - 2] \setminus \{2n - 118, 2n - 6, 2n - 2\}.
\]
Hence for $120 \leq n \leq 178$ we have that $A \uparrow A$ contains
\[
[0, 2n - 2] \setminus \{0, 1, 4, 6, 8, 15, 22, 29, 36, 43, 50, 120, 2n - 118, 2n - 6, 2n - 2\}
\]
so that $|A \uparrow A| \geq 2n - 16$. There are $n - 120$ numbers between 61 and $n - 60$ inclusive. Therefore the number of such $A$ is $2^{n-120}$.

For $n \geq 178$ applying Proposition 16 with $\ell = 61$ and $u = 59$ implies that when $A$ is chosen uniformly randomly from all such sets, the probability that $A \uparrow A$ contains $[61, n - 60] \cup [n + 60, 2n - 119]$ is at least
\[
1 - 8(2^{-|L|} + 2^{-|U|}) = 1 - 8(2^{-29} + 2^{-35}) = \frac{4294967231}{4294967296}.
\]
That is, there are at least $2^{n-120} \frac{4294967231}{4294967296} > (7.52 \times 10^{-37})2^n$ such sets $A$ with $A \uparrow A = [0, 2n - 2] \setminus \{0, 1, 4, 6, 8, 15, 22, 29, 36, 43, 50, 120, 2n - 118, 2n - 6, 2n - 2\}$, whilst at the same time eight pairs of differences are excluded from $A - A$. Thus all such sets $A$ are restricted-sum-dominant sets. 

Martin and O’Bryant’s Lemma 7 and Theorem 16 for a subset $S$ of an arithmetic progression of length $n$ can also be adapted to give the following result.

**Theorem 18.** Given a subset $S$ of an arithmetic progression $P$ of length $n$ for every positive integer $n$, we have
\[
\sum_{S \subseteq P} |S \uparrow S| = 2^n(2n - 15) + \begin{cases} 26 \cdot 3^{(n-1)/2}, & \text{if } n \text{ is odd}, \\ 15 \cdot 3^{n/2}, & \text{if } n \text{ is even}. \end{cases}
\]

Thus $\frac{1}{|P|} \sum_{S \subseteq P} |S \uparrow S| \sim 2n - 15$. This combined with Martin and O’Bryant’s Theorem 3, that $\frac{1}{|P|} \sum_{S \subseteq P} |S - S| \sim 2n - 7$ gives that on average the difference set has eight elements more than the restricted subset. Details will appear in [10].
4. How Much Larger Can the Sun Set Be?

As in Section 4 of [3] we consider this question in terms of \( f(A) = \ln|A+A|/\ln|A-A| \) (and the analogous quantity \( f(A) = \ln|A+A|/\ln|A-A| \)). It is known – see, e.g., [1] – that \( \frac{2}{3} \leq f(A) \leq \frac{4}{3} \). The reason for considering the ratio of logarithms rather than (say) the ratio is explained in [3] in terms of the base expansion method. Some authors, e.g., Granville in [2], prefer to use \( g(A) = \ln(|A+A|)/\ln(|A-A|) \) for which the analogous bounds are \( 1/2 \leq g(A) \leq 2 \).

Hegarty’s set \( A_{15} \) is easily checked to have \( f(A_{15}) = 1.0208 \ldots \), which is often quoted as the largest known value of \( f(A) \). In fact, the set \( X \) (our \( T_2 \)) which Hegarty uses to write \( A_{15} = X \cup (X + 20) \) already does fractionally better:

**Lemma 19.** Let \( X = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25\} \). Then \( X + X = [0, 50] \) but \( X - X = [-25, 25] \setminus \{\pm 6, \pm 14\} \). Thus \( f(X) = \ln(51)/\ln(47) \simeq 1.0212 \).

**Proof.** This is just a short calculation.

We do better than either of these using the sets \( Q_j \) at the end of Section 2.

**Theorem 20.** There is a set \( A \) of integers for which

\[
f(A) = \frac{\ln(|A+A|)}{\ln(|A-A|)} \simeq 1.030597781 \ldots
\]

and another set \( B \) of integers for which

\[
\hat{f}(B) = \frac{\ln(|B+B|)}{\ln(|B-B|)} \simeq 1.028377107 \ldots
\]

**Proof.** Take \( A = Q_{10} \) for the first claim and \( A = Q_{19} \) for the second claim.

It is easy to check that neither any other \( Q_j \), nor any of the \( T_j, T'_j, M_j \) or \( R_j \) give better results than the two \( Q_j \)'s listed above.

The function \( g \) has a slightly different behaviour, as it is monotone increasing as \( j \) increases in our sequences. The result here is

**Theorem 21.** Given \( \varepsilon > 0 \), there is a set \( C \) of integers for which

\[
g(C) = \frac{\ln(|C+C|/|C|)}{\ln(|C-C|/|C|)} > \frac{\ln(32/5)}{\ln(26/5)} - \varepsilon \simeq 1.125944426
\]

**Proof.** Take \( Q_j \) for \( j \) sufficiently large.

(For comparison, \( g(A_{15}) \simeq 1.0717 \). The corresponding suprema are \( \ln(16/3)/\ln(14/3) \simeq 1.0867 \) for both \( (g(T'_j)) \) and \( (g(T_j)) \), \( \ln(23/4)/\ln(11/2) \simeq 1.0261 \) for \( (g(R_j)) \) and \( \ln(11/2)/\ln(5) \simeq 1.0592 \) for \( (g(M_j)) \). None of these do as well as the supremum for the \( (Q_j) \).)
Note also that because the sumsets and restricted sumsets in each of our families $T_j', T_j, M_j, R_j$ and $Q_j$ only differ in order by a constant, the function

$$\hat{g}(A) = \frac{\ln(|A-A|/|A|)}{\ln(|A-A|/|A|)}$$

will give similar insights to $g$.

5. The Smallest Order of a Restricted-Sum-Dominant Set

We noted above that we have two restricted-sum-dominant sets of order 16, namely $T_2'$ and $M_2$: we know of no smaller examples. In this section we reduce the range in which the smallest restricted-sum-dominant set can be.

Hegarty ([3], Theorem 1) proves that no 7-element subset of the integers is sum-dominant, and that up to linear transformations Conway’s set is the unique 8-element sum-dominant subset of $\mathbb{Z}$. As Conway’s set is not a restricted-sum-dominant set there is no 8-element restricted-sum-dominant set of integers.

Further Hegarty finds all 9-element sum-dominant sets $A$ of integers with the additional property that for some $x \in A+A$ there are at least four ordered pairs $(a,a') \in A \times A$ with $a+a' = x$. There are, up to linear transformations, 9 such sets, listed in [3] as $A_2$ and $A_4$ through to $A_{11}$. It is easy to check that none of these nine sets is restricted-sum-dominant.

Thus, the only possible 9-element restricted-sum-dominant sets of integers have the property that for every $x \in A+A$ there are fewer than four ordered pairs $(a,a')$ such that $x = a+a'$. This condition implies that there is no solution of $x+y = u+v$ with $x,y,u,v$ all distinct, so such a set is a weak Sidon set in the sense of Ruzsa [8].

Defining $\delta(n)$ for $n \in A-A$ to be the number of ordered pairs $(x,y)$ such that $x+y = n$, it is shown in the proof of Theorem 4.7 in [8] that for a weak Sidon set, $\delta(n) \leq 2$ whenever $n \neq 0$ and at most $2|A|$ elements $n$ have $\delta(n) = 2$.

Thus, noting 0 has $|A| = 9$ representations and putting $m = |A-A|$, \[81 \leq 9 + (2 \times 9) \times 2 + (m - 19) \Rightarrow m \geq 55\]

so if such a set were to be sum-dominant its sumset would have to have order at least 56. But of course $|A+A| \leq 9 \times 10/2 = 45$, and we have proven

**Theorem 22.** All sum-dominant sets of integers of order 9 are linear transformations of one of Hegarty’s nine sets $A_2$ and $A_4$ to $A_{11}$. None of these is restricted-sum-dominant, so there is no restricted-sum-dominant set of order 9.

We thus know that the smallest restricted-sum-dominant set of integers has order between 10 and 16. It appears a non-trivial computational challenge to find the order of the smallest restricted-sum-dominant set.
References


