GENERALIZATION OF UNIVERSAL PARTITION AND
BIPARTITION THEOREMS

Hacène Belbachir\textsuperscript{1}
\textit{USTHB, Faculty of Mathematics, RECITS Laboratory, Algiers, Algeria}
hbelbachir@usthb.dz, hacenebelbachir@gmail.com

Miloud Mihoubi\textsuperscript{2}
\textit{USTHB, Faculty of Mathematics, RECITS Laboratory, Algiers, Algeria.}
mmihoubi@usthb.dz, miloudmihoubi@gmail.com

Received: 5/24/12, Revised: 5/4/13, Accepted: 8/10/13, Published: 9/26/13

Abstract

Let $A = (a_{i,j})$, $i = 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, be an infinite matrix with elements $a_{i,j} = 0$ or 1; $p(n, k; A)$ the number of partitions of $n$ into $k$ parts whose number $y_i$ of parts which are equal to $i$ belongs to the set $Y_i = \{j : a_{i,j} = 1\}$, $i = 1, 2, \ldots$. The universal theorem on partitions states that

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{n} p(n, k; A) u^k t^n = \prod_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} a_{i,j} u^j t^j \right).
$$

In this paper, we present a generalization of this result. We show that this generalization remains true when $a_{i,j}$ are indeterminate. We also take into account the bi-partite and multi-partite situations.

1. Introduction

Let $A = (a_{i,j})$, $i = 1, 2, \ldots$, $j = 0, 1, 2, \ldots$ be an infinite matrix with elements $a_{i,j} = 0$ or 1; $p(n, k; A)$ the number of partitions of $n$ into $k$ parts whose number $y_i$ of parts which are equal to $i$ belongs to the set $Y_i = \{j : a_{i,j} = 1\}$, $i = 1, 2, \ldots$. The universal theorem on partitions states that

$$
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} p(n, k; A) u^k \right) t^n = \prod_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} a_{i,j} u^j t^j \right);
$$

(1)

\textsuperscript{1}The research is partially supported by the LITIS Laboratory of Rouen University and the PNR project 8/u160/664.

\textsuperscript{2}This research is supported by the PNR project 8/u160/3172.
see for instance [2] and [3].

In Section 2, we will provide an extension of the above identity and show that it remains true when \( a_{i,j} \) are indeterminate. In Section 3, we will present an equivalent version in terms of complete Bell polynomials when \( a_{i,0} = 1, i \geq 1 \).

Similarly, a partition of an ordered pair \((m, n) \neq (0, 0)\), of nonnegative integers, is a non-ordered collection of nonnegative integers \((x_i, y_i) \neq (0, 0), i = 1, 2, \ldots\), whose sum equals \((m, n)\). Given a partition of \((m, n)\), let \(k_{i,j}\) be the number of parts which are equal to \((i, j)\), \(i = 0, 1, 2, \ldots, m, j = 0, 1, 2, \ldots, n, (i, j) \neq (0, 0)\), such that

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} k_{i,j} = m, \quad \sum_{j=0}^{n} \sum_{i=0}^{m} k_{i,j} = n. \tag{2}
\]

For a partition of \((m, n)\) into \(k\) parts, we add

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} k_{i,j} = k. \tag{3}
\]

Let \(p(m, n)\) be the number of partitions of the bi-partite number \((m, n)\) with \(p(0, 0) = 1\) and \(p(m, n, k)\) be the number of partitions of \((m, n)\) into \(k\) parts with \(p(0, 0, 0) = 1\). The universal bipartition theorem states that

\[
F(t, u, w) = \sum_{m,n \geq 0} \left( \sum_{k=0}^{m+n} p(m, n, k) w^k \right) t^m u^n = \prod_{j=0}^{\infty} \prod_{i=0}^{\infty} \left( 1 - wt^i u^j \right)^{-1}; \tag{4}
\]

see [2, p. 403, pb. 24]. A generalization of identity (4) is dealt with in Section 4. Section 5 is devoted to the concept of multipartition.

2. Generalized Universal Partition Theorem

**Theorem 1.** Let \(X = (x_{i,j}), i = 1, 2, \ldots, j = 0, 1, 2, \ldots\), be an infinite matrix of indeterminates; \(\pi(n, k)\) the set of all nonnegative integer solutions of

\[
k_1 + k_2 + \cdots + k_n = k \quad \text{and} \quad k_1 + 2k_2 + \cdots + nk_n = n;
\]

and \(\pi(n) = \bigcup_{k=1}^{n} \pi(n, k)\) be the set of all nonnegative integer solutions of \(k_1 + 2k_2 + \cdots + nk_n = n\). For every solution \(k_1, k_2, \ldots, k_n\), we set

\[
p(n, k; X) := \sum_{\pi(n, k)} x_{1,k_1} x_{2,k_2} \cdots x_{n,k_n}.
\]

Then

\[
G(t, u; X) := \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} p(n, k; X) u^k \right) t^n = \prod_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} x_{i,j} u^j t^i \right).
\]
Proof. We have

\[ G(t, u; X) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \sum_{\pi(n, k)} x_{1,k_1} x_{2,k_2} \cdots x_{n,k_n} u^{k_1+\cdots+k_n} t^{k_1+2k_2+\cdots+nk_n} \right). \]

Since these sums apply for all \( k = 0, 1, \ldots, n \) and \( n = 0, 1, \ldots \), it follows that

\[ G(t, u; X) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \sum_{\pi(n, k)} \left( x_{1,k_1} (ut)^{k_1} \right) \left( x_{2,k_2} (ut^2)^{k_2} \right) \cdots \left( x_{n,k_n} (ut^n)^{k_n} \right) \right) \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{\pi(n)} x_{1,k_1} (ut)^{k_1} \right) \left( \sum_{k_2=0}^{\infty} x_{2,k_2} (ut^2)^{k_2} \right) \cdots \]

\[ = \prod_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} x_{i,j} (ut^i)^j \right), \]

which is the required expression. \( \square \)

For \( x_{i,j} = a_{i,j} \) with \( i = 1, 2, \ldots, j = 0, 1, 2, \ldots \), in Theorem 1, we obtain the universal theorem on partitions. For \( x_{i,j} = \frac{a_{i,j}}{n!} \) with \( i = 1, 2, \ldots, j = 0, 1, 2, \ldots \), in Theorem 1, we obtain:

**Corollary 2.** Let \( A = (a_{i,j}) \), \( i = 1, 2, \ldots, j = 0, 1, 2, \ldots \), be an infinite matrix with elements \( a_{i,j} = 0 \) or 1 and \( c(n, k; A) \) the number of permutations of a finite set \( W_n \), of \( n \) elements, that are decomposed into \( k \) cycles such that the number of cycles of length \( i \) belongs to the set \( Y_i = \{ j : a_{i,j} = 1 \} \). Then

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} c(n, k; A) u^k t^n \frac{n!}{n!} = \prod_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} a_{i,j} \frac{y^j}{j!} \left( t^i \right)^j \right). \]

For \( x_{i,j} = \frac{1}{\pi} \left( \frac{2\pi}{i} \right)^j a_{i,j} \), \( z_i \in \mathbb{C} \), \( i = 1, 2, \ldots, j = 0, 1, 2, \ldots \), in Theorem 1, we obtain a remarkable identity according to the partial Bell polynomials:

**Corollary 3.** Let \( A = (a_{i,j}) \), \( i = 1, 2, \ldots, j = 0, 1, 2, \ldots \), be an infinite matrix with elements \( a_{i,j} = 0 \) or 1 and

\[ B_{n,k;A}(z_1, z_2, \ldots, z_n) := \sum_{\pi(n, k)} \frac{n!}{k_1! \cdots k_n!} \left( \frac{z_1}{1!} \right)^{k_1} \cdots \left( \frac{z_n}{n!} \right)^{k_n} a_{1,k_1} \cdots a_{n,k_n}. \]
Then we obtain
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} B_{n,k;A} (z_1, z_2, \ldots, z_n) \frac{t^n}{n!} = \prod_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} a_{i,j} \left( \frac{u z_i t^i}{i!} \right)^j \right).
\]

**Remark 4.** For \(a_{i,j} = 1, i = 1, 2, \ldots\) and \(j = 0, 1, \ldots\), Corollary 3 gives
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} B_{n,k} (z_1, z_2, \ldots, z_n) \frac{t^n}{n!} = \exp \left( u \sum_{i=1}^{\infty} z_i t^i \right),
\]
which is the definition of the partial Bell polynomials \(B_{n,k} (z_1, \ldots, z_n)\). See [1, 3, 4].

3. **Connection With the Complete Bell Polynomials**

Recall that the complete Bell polynomials \(A_n (x_1, x_2, \ldots)\) are defined by
\[
\sum_{n=0}^{\infty} A_n (x_1, x_2, \ldots) \frac{t^n}{n!} = \exp \left( \sum_{m=1}^{\infty} x_m t^m \right).
\]
See [1, 3, 4].

In this section, we provide another formulation of Theorem 1 according to the complete Bell polynomials. We determine the generating functions of the sequences \((p(n,k;X))_n\) and \((p(n,k;X))_k\), where \(X = (x_{i,j})\), \(i = 1, 2, \ldots, j = 0, 1, 2, \ldots\), is an infinite matrix with indeterminates \(x_{i,j}\) such that \(x_{i,0} = 1\) for every \(i \geq 1\).

**Theorem 5.** Let \(q, u\) be indeterminate. Then, for \(n \geq 1\), we have
\[
\sum_{j=0}^{n} p(n,j;X) u^j = \frac{1}{n!} A_n (\rho_1 (q;X), \rho_2 (q;X), \ldots, \rho_n (q;X)), \quad (5)
\]
\[
\sum_{j=0}^{n} p(j,n;X) q^j = \frac{1}{n!} A_n (\sigma_1 (u;X), \sigma_2 (u;X), \ldots, \sigma_n (u;X)), \quad (6)
\]
where
\[
\rho_n (q;X) := \sum_{i=1}^{\infty} b_n (i) q^{ni} \quad \text{and} \quad \sigma_n (u;X) := n! \sum_{k|n} b_k \left( \frac{n}{k} \right) u^k,
\]
with
\[
b_n (i) = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k} (1! x_{i,1}, 2! x_{i,2}, \ldots, j! x_{i,j}, \ldots).
\]
Proof. From Theorem 1, we get

\[ G(q, u; X) := \sum_{n=0}^{\infty} \sum_{k=0}^{n} p(n, k; X) u^k q^n = \exp \left( \sum_{i=1}^{\infty} \ln \left( 1 + \sum_{j=1}^{\infty} (j! x_{i,j}) \frac{(uq^i)^j}{j!} \right) \right). \]

Using the following known expansion (see [2, Theorem 11.17])

\[ \ln \left( 1 + \sum_{k=1}^{\infty} g_k q^k \right) = \sum_{n=1}^{\infty} \frac{c_n q^n}{n!}, \]

with \( c_n = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k} (g_1, g_2, \ldots) \), we obtain

\[ G(q, u; X) = \exp \left( \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} b_k (i) \frac{u^k}{k!} q^{ki} \right) = \exp \left( \sum_{k=1}^{\infty} \frac{u^k}{k!} \sum_{i=1}^{\infty} b_k (i) q^{ki} \right) = \exp \left( \sum_{k=1}^{\infty} \rho_k (q; X) \frac{u^k}{k!} \right) = 1 + \sum_{k=1}^{\infty} A_k (\rho_1 (q; X), \ldots, \rho_k (q; X)) \frac{u^k}{k!}. \]

On the other hand, we have

\[ G(q, u; X) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} p(n, k; X) u^k q^n = \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} p(n, k; X) q^n \right) u^k. \]

The first identity follows from identification, where as the second identity follows from the expansion

\[ G(q, u; X) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} p(n, k; X) u^k \right) q^n = \exp \left( \sum_{i=1}^{\infty} \ln \left( 1 + \sum_{j=1}^{\infty} (j! x_{i,j}) \frac{(uq^i)^j}{j!} \right) \right) = \exp \left( \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} b_k (i) \frac{u^k}{k!} q^{ki} \right) = \exp \left( \sum_{n=1}^{\infty} q^n \sum_{k|n} b_k \left( \frac{n}{k} \right) \frac{u^k}{k!} \right) = \exp \left( \sum_{n=1}^{\infty} \sigma_n (u; X) \frac{q^n}{n!} \right) = 1 + \sum_{n=1}^{\infty} A_n (\sigma_1 (u; X), \sigma_2 (u; X), \ldots, \sigma_n (u; X)) \frac{q^n}{n!}. \]

\[ \square \]
Corollary 6. Let $\alpha$ and $q$ be two indeterminates. We then have

\[
A_n \left( \frac{1}{1-q} - \alpha, \ldots, (n-1)! \left( \frac{1}{1-q^n} - \alpha^n \right) \right) = n! \left( \frac{1}{1-q^n} - \alpha \right) \prod_{i=1}^{n-1} (1-q^i)^{-1}
\]

and

\[
A_n \left( (1-\alpha) u, \ldots, (n-1)! \left( \sum_{k|n} k u^{n/k} - (\alpha u)^n \right) \right) = n! (p_n (u) - \alpha u p_{n-1} (u)),
\]

where $p_n (u) := \sum_{j=0}^{n} p(n, j) u^j$.

Proof. We put in identity (5) $x_{1,0} = 1$, $x_{1,j} = q^{-j} (1-\alpha)$ for $j \geq 1$ and $x_{i,j} = q^{-j}$ for $i \geq 2$, $j \geq 0$, and use identity $B_{n,k} (1!, 2!, 3!, \ldots) = \binom{n-1}{k-1}! \binom{n}{k}$ (Lah numbers). We obtain

\[
b_n (1) = (n-1)! (1-\alpha^n) q^{-n}, \quad b_n (i) = (n-1)! q^{-n}, \quad i \geq 2,
\]

\[
\rho_n (q; X) = \sum_{i=1}^{\infty} b_n (i) q^{ni} = (n-1)! \left( \frac{1}{1-q^n} - \alpha^n \right),
\]

\[
p(n, k; X) = q^{-k} \sum_{\pi(n,k), k_1=0}^{\infty} 1 + q^{-k} \sum_{\pi(n,k), k_1 \geq 1} (1-\alpha)
\]

\[
= q^{-k} \sum_{\pi(n-k,k)} 1 + (1-\alpha) q^{-k} \sum_{\pi(n-1,k-1)} 1
\]

\[
= q^{-k} \left[ p(n-k, k) + (1-\alpha) p(n-1, k-1) \right],
\]

where $p(n, k)$ is the number of partitions of $n$ into $k$ parts, which satisfy

\[
p(n, k) = p(n-k, k) + p(n-1, k-1).
\]

Thus, we obtain

\[
\sum_{j=n}^{\infty} p(j, n; X) q^j = q^{-n} \left( \sum_{j=n}^{\infty} p(j, n) q^j - \alpha \sum_{j=n}^{\infty} p(j-1, n-1) q^j \right)
\]

\[
= \left( \frac{1}{1-q^n} - \alpha \right) \prod_{i=1}^{n-1} (1-q^i)^{-1},
\]

which gives the first identity.

For the second identity, we take $x_{1,0} = 1$, $x_{1,j} = 1 - \alpha$ for $j \geq 1$ and $x_{i,j} = 1$ for
\[i \geq 2, \, j \geq 0,\] in relation (6) to get
\[b_n (1) = -(n-1)! \left( \alpha^n - 1 \right), \quad b_n (i) = (n-1)! , \quad i \geq 2,\]
\[\sigma_n (u; X) = n! \sum_{k \mid n} b_k \left( \frac{n}{k} \right) \frac{u^k}{k!} = -(n-1)! \left( \alpha u \right)^n + (n-1)! \sum_{k \mid n} ku^{n/k},\]
\[p (n, k; X) = p (n, k) - \alpha p (n-1, k-1),\]
thus
\[\sum_{j=0}^{n} p (n, j; X) u^j = p_n (u) - \alpha up_{n-1} (u),\]
which provides the second identity.

4. Generalized Universal Bipartition Theorem

In this section, we provide a generalization of identity (4) and deduce some known identities. Let us start with the following example: how do we partition \((2, 3)\) into different parts? Let \(p (2, 3, k)\) be the number of partitions of the bi-partite number \((2, 3)\) into \(k\) parts, \(k = 1, \ldots, 5\) and \(p (2, 3)\) be the total number of partitions of \((2, 3)\). We have
\[(2, 3)\]
\[p (2, 3, 1) = 1\]
\[p (2, 3, 2) = 5\]
\[p (2, 3, 3) = 5\]
\[p (2, 3, 4) = 3\]
\[p (2, 3, 5) = 1\]
\[p (2, 3) = \sum_{k=1}^{5} p (2, 3, k) = 1 + 5 + 5 + 3 + 1 = 15.\]

**Theorem 7.** Let \(X = (x_{i,j,s})\), \(i,j,s = 0,1,2,\ldots,\) be a sequence of indeterminates with \(x_{0,0,s} = 0\), \(\Pi (m, n, k)\) the set of all nonnegative integers \(k_{i,j}\) satisfying (2) and (3) and \(\Pi (m, n) := \bigcup_{k=1}^{m+n} \Pi (m, n, k)\) the set of all nonnegative integers satisfying (2). For every partition of the bi-partite number \((m, n)\) into \(k\) parts, we set
\[p (m, n, k; X) := \sum_{\Pi (m, n, k)} \prod_{i=0}^{m} x_{i,j,k_{i,j}},\]
Then we have

\[ F(t, u, \omega; X) := \sum_{m,n \geq 0} \left( \sum_{k=0}^{m+n} \binom{m+n}{k} \left( \sum_{i=0}^{m} \sum_{j=0}^{n} x_{i,j,k_i,j} \omega^{i+j} \right) \right) t^m u^n = \prod_{i+j \geq 1} \left( \sum_{s=0}^{\infty} x_{i,j,s} \left( \omega^{i+j} \right)^s \right). \]

**Proof.** We have

\[
F(t, u, \omega; X) = \sum_{m,n \geq 0} \left( \sum_{k=0}^{m+n} \binom{m+n}{k} \left( \sum_{i=0}^{m} \sum_{j=0}^{n} x_{i,j,k_i,j} \omega^{i+j} \right) \right) t^m u^n \\
= \sum_{m,n \geq 0} \sum_{k=0}^{m+n} \left( \prod_{i} \prod_{j} \prod_{k_i,j=0}^{m} \prod_{i=0}^{m} \prod_{j=0}^{n} x_{i,j,k_i,j} \omega^{i+j} \right) t^m u^n \\
= \prod_{i+j \geq 1} \left( \sum_{k_i,j=0}^{\infty} x_{i,j,k_i,j} \left( \omega^{i+j} \right)^{k_i,j} \right). 
\]

**Corollary 8.** Let \( A = (a_{i,j,s}) \), \( i, j, s = 0, 1, 2, \ldots \), with \( a_{i,j,s} = 0 \) or \( 1 \) for \( (i, j) \neq (0, 0) \) and let \( p(m, n, k; A) \) be the number of partitions of \( (m, n) \) into \( k \) parts whose number \( y_{i,j} \) of parts which are equal to \( (i, j) \) belongs to the set \( Y_{i,j} = \{s : a_{i,j,s} = 1\} \), \( i, j = 0, 1, 2, \ldots, (i, j) \neq (0, 0) \). Then

\[ F(t, u, \omega; A) := \sum_{m,n \geq 0} \left( \sum_{k=0}^{m+n} \binom{m+n}{k} \left( \sum_{i=0}^{m} \sum_{j=0}^{n} x_{i,j,k_i,j} \omega^{i+j} \right) \right) t^m u^n = \prod_{i+j \geq 1} \left( \sum_{s=0}^{\infty} a_{i,j,s} \left( \omega^{i+j} \right)^s \right). \]

For \( x_{i,j,s} = 1 \) for all \( i, j, s = 0, 1, 2, \ldots, (i, j) \neq (0, 0) \), Theorem 7 becomes:

**Corollary 9.** Let \( p(m, n) \) be the number of partitions of the bi-partite number \( (m, n) \) with \( p(0, 0) = 1 \) and \( p(m, n, k) \) the number of partitions of \( (m, n) \) into \( k \) parts, with \( p(0, 0, 0) = 1 \). Then

\[ \sum_{m,n \geq 0} \left( \sum_{k=0}^{m+n} \binom{m+n}{k} \left( \sum_{i=0}^{m} \sum_{j=0}^{n} x_{i,j,k_i,j} \omega^{i+j} \right) \right) t^m u^n = \prod_{i+j \geq 1} \left( 1 - \omega^{i+j} \right)^{-1}. \]

**Remark 10.** Let \( (y_{i,j}) \), \( i, j = 0, 1, \ldots \), be a sequence of indeterminates and let \( x_{i,j,s} = \frac{1}{s!} \left( \frac{y_{i,j}}{i!j!} \right)^s \), \( i, j = 0, 1, 2, \ldots \), we have

\[ p(m, n, k; X) = \sum_{\Pi(m,n,k)} \prod_{i=0}^{m} \prod_{j=0}^{n} \frac{1}{k_{i,j}!} \left( \frac{y_{i,j}}{i!j!} \right)^{k_{i,j}} = A_{m,n,k} \frac{m!n!}{m!n!}, \]
where
\[ A_{m,n,k} := A_{m,n,k}(y_{0,1}, y_{1,0}, \ldots, y_{m,n}), \]
and
\[ F(t, u, \omega; X) = \sum_{m,n \geq 0} \left( \sum_{k=0}^{m+n} A_{m,n,k} \omega^k \right) \frac{t^m u^n}{m! \ n!}. \]

From Theorem 7, we obtain
\[ F(t, u, \omega; X) = \prod_{i+j \geq 1} \left( \sum_{k_i,j \geq 0} \frac{1}{k_i,j!} \left( \omega y_{i,j} t^i u^j \right)^{k_i,j} \right) = \exp \left( \omega \left( \sum_{i+j \geq 1} y_{i,j} t^i u^j \right) \right). \]

From the two expressions of \( F(t, u, \omega; X) \), we retrieve the exponential partial bi-partitional polynomials:
\[ \sum_{m,n \geq 0} \left( \sum_{k=0}^{m+n} A_{m,n,k}(y_{0,1}, y_{1,0}, \ldots, y_{m,n}) \omega^k \right) \frac{t^m u^n}{m! \ n!} = \exp \left( \omega \left( \sum_{i+j \geq 1} y_{i,j} t^i u^j \right) \right); \]
see [2, pp. 454–457].

5. Universal Multipartition Theorem

More generally, a multipartition of order \( r \) of \( n = (n_1, \ldots, n_r) \), different from \( 0 = (0, \ldots, 0) \), of nonnegative integers, is a non-ordered collection of nonnegative integers \((x_i^{(1)}, \ldots, x_i^{(r)})\), \( i = 1, 2, \ldots \), whose sum equals \( n \). In a partition of an \( r \)-partite number \( n \), let \( k_i := k_{i_1, \ldots, i_r} \) be the number of ordered \( r \) numbers that are equal to \( i = (i_1, \ldots, i_r) \in \{0, 1, 2, \ldots, n_1\} \times \cdots \times \{0, 1, 2, \ldots, n_r\} \), \((i_1, \ldots, i_r) \neq 0 \), such that
\[ \sum_{i_1=0}^{n_1} \cdots \sum_{i_r=0}^{n_r} i_j k_{i_1, \ldots, i_r} = n_j, \quad j = 1, \ldots, r. \]  
(7)

For the partition of \( n \) into \( k \) parts, we add
\[ \sum_{i_1=0}^{n_1} \cdots \sum_{i_r=0}^{n_r} k_{i_1, \ldots, i_r} = k. \]  
(8)

Let \( p(n) \) be the number of partitions of the \( r \)-partite \( n \) with \( p(0) = 1 \) and \( p(n, k) \) the number of partitions of the \( r \)-partite number \( n \) into \( k \) parts, with \( p(0, 0) = 1 \).

**Theorem 11.** Let \( X = (x_{i,s}), \ i = (i_1, \ldots, i_r) \in \mathbb{N}^r, \ i \neq 0, \ s = 0, 1, 2, \ldots, \) be a sequence of indeterminates with \( r + 1 \) indices, with \( x_{0,s} = 0, \) \( \Pi(n,k) \) the set of all nonnegative integers \( k_{i_1,\ldots,i_r} \) satisfying (7) and (8) and \( \Pi(n) := \bigcup_{k=1}^{n_1+\cdots+n_r} \Pi(n,k) \).
the set of all nonnegative integers solutions of (7). For every partition of \( n \) into \( k \) parts, we set

\[
p(n; k, X) := \sum_{\Pi(n, k)} \prod_{i=0}^{n} x_{i, k_i}.
\]

Then

\[
F(t, \omega; X) = \sum_{n \geq 0} \left( \sum_{k=0}^{n-1} p(n; k, X) \omega^k \right) t^n = \prod_{i \geq 1} \left( \sum_{s=0}^{\infty} \sum_{x_{i, k_i} \geq 0} x_{i, k_i} (\omega t^i)^{k_i} \right).
\]

where \( t^n := t_1^{n_1} \cdots t_r^{n_r} \), \( n \cdot 1 \equiv n_1 + \cdots + n_r, n \geq 0 \Leftrightarrow n_1 \geq 0, \ldots, n_r \geq 0 \).

Proof. We have the following

\[
F(t, \omega; X) = \sum_{n \geq 0} \left( \sum_{k=0}^{n-1} p(n; k, X) \omega^k \right) t^n = \sum_{n \geq 0} \sum_{k=0}^{n-1} \left( \sum_{\Pi(n, k)} \prod_{i=0}^{n} x_{i, k_i} (\omega t^i)^{k_i} \right) = \sum_{n \geq 0} \sum_{\Pi(n)} \prod_{i=0}^{n} x_{i, k_i} (\omega t^i)^{k_i},
\]

and exploiting \( x_{0,0} = 0 \), the last expression becomes

\[
\prod_{i \geq 0} \left( \sum_{k_i \geq 0} x_{i, k_i} (\omega t^i)^{k_i} \right) = \prod_{i \geq 1} \left( \sum_{k_i \geq 0} x_{i, k_i} (\omega t^i)^{k_i} \right).
\]

For \( x_{i,s} = a_{i,s} \in \{0,1\} \) for all \( i \in \mathbb{N}^r, i \neq 0, s = 0, 1, 2, \ldots \), we obtain:

**Corollary 12.** Let \( A = (a_{i,s}), i \in \mathbb{N}^r, i \neq 0, s = 0, 1, 2, \ldots \), with \( a_{i,s} = 0 \) or \( 1 \) and \( p(n, k; A) \) be the number of partitions of \( n \) into \( k \) parts whose number \( y_i \) of parts which are equal to \( i \) belongs to the set \( Y_i = \{ s : a_{i,s} = 1 \} \), \( i \in \mathbb{N}^r, i \neq 0 \). Then

\[
F(t, u, \omega; A) := \sum_{n \geq 0} \left( \sum_{k=0}^{n-1} p(n, k; A) \omega^k \right) t^n = \prod_{i \geq 1} \left( \sum_{s \geq 0} a_{i,s} (\omega t^i)^s \right).
\]

For \( x_{i,s} = 1 \) for all \( i \in \mathbb{N}^r, i \neq 0, s = 0, 1, 2, \ldots \), we obtain:
Corollary 13. Let $p(n)$ be the number of partitions of the $r$-partite number $n$ with $p(0) = 1$ and $p(n,k)$ the number of partitions of the $r$-bipartite number $n$ into $k$ parts, with $p(0,0) = 1$. Then
\[
\sum_{n \geq 0} \left( \sum_{k=0}^{n-1} p(n,k) \omega^k \right) t^n = \prod_{i \geq 1} \left( 1 - \omega^i \right)^{-1}.
\]
Consequently
\[
\sum_{n \geq 0} p(n) t^n = \prod_{i \geq 1} \left( 1 - t^i \right)^{-1}.
\]
Remark 14. If we take $t_1 = \cdots = t_r = t$, we obtain
\[
\prod_{i \geq 1} \left( 1 - \omega^i \right)^{-\binom{r+1}{r-1}} = \sum_{n \geq 0} \sum_{n_1 + \cdots + n_r = n} p(n,k) \omega^n t^n = \sum_{n \geq 0} \left( \sum_{n_1 + \cdots + n_r = n} p_n(\omega) \right) t^n,
\]
and more generally, for nonnegative integers $a_1, \ldots, a_r$ and $t_1 = t^{a_1}, \ldots, t_r = t^{a_r}$, we obtain
\[
\prod_{i \geq 1} \left( 1 - \omega^i \right)^{-f(i,r)} = \sum_{n \geq 0} \left( \sum_{a_1 n_1 + \cdots + a_r n_r = n} p_n(\omega) \right) t^n,
\]
where
\[
p_n(\omega) = \sum_{k=0}^{a_1 n_1 + \cdots + a_r n_r} p(n,k) \omega^k,
\]
and where $f(n,r)$ is the number of solutions of the integer equation
\[
a_1 n_1 + a_2 n_2 + \cdots + a_r n_r = n.
\]

Acknowledgments The authors wish to express their gratitude to the referee for his/her valuable advice and comments which helped to greatly in improving the quality of the paper.

References