EIGENVALUES AND ARITHMETIC FUNCTIONS ON PSL$_2(\mathbb{Z})$

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Abstract
Over the past decade, various properties of the irrational factor function $I(n) = \prod_{p^r || n} p^{1/r}$ and strong restrictive factor function $R(n) = \prod_{p^r || n} p^{r-1}$ have been investigated by several authors. This study led to a generalization to a class of arithmetic functions associated to elements of PSL$_2(\mathbb{Z})$. In the present paper, we study the possible influence of the eigenvalues of an element $A$ of PSL$_2(\mathbb{Z})$ on the behavior of the associated arithmetic function $f_A(n) = \prod_{p^r || n} p^{A(r)}$, where $A(z) = (az+b)/(cz+d)$ is the linear fractional transformation induced by the matrix $A$. In particular, we obtain results on the local density of eigenvalues through their natural connection to a particular surface.

1. Introduction and Statement of Results

There has been recent interest in examining the behavior of the arithmetic functions $f_A(n)$ defined on natural numbers $n$ in terms of the action of a matrix $A$ in PSL$_2(\mathbb{Z})$. Given an element

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
of $\text{PSL}_2(\mathbb{Z})$, one may consider the linear fractional transformation induced by $A$,

$$A(z) = \frac{az + b}{cz + d},$$

and define the arithmetic function given for each positive integer $n$ by

$$f_A(n) = \prod_{p^\nu || n} p^{A(\nu)}.$$

These functions generalize the two arithmetic functions

$$I(n) = \prod_{p^\nu || n} p^{1/\nu}$$

and

$$R(n) = \prod_{p^\nu || n} p^{\nu - 1},$$

which were introduced by Atanassov in [2] and [3]. These multiplicative functions satisfy the inequality

$$I(n)R(n)^2 \geq n,$$

for each $n \geq 1$, with equality if and only if $n$ is square-free. If $S(n)$ denotes the square-free part of $n$ and if $n$ is $k$-power free, then $S(n)$ satisfies the inequalities

$$S(n) \geq n^{1/(k-1)}$$

and

$$I(n) \geq S(n)^{1/(k-1)} \geq n^{1/(k-1)^2}.$$  

On the other hand, if $n$ is $k$-power full, then $S(n)$ satisfies the inequality

$$I(n) \leq S(n)^{1/k}.$$  

In this fashion, $I(n)$ roughly measures how far a given integer $n$ is away from being either $k$-power free or $k$-power full.

In [11], two of the authors more fully develop this measure by studying weighted combinations $I(n)^\alpha R(n)^\beta$ for real-valued $\alpha$ and $\beta$. In [10], Panaitopol showed that

$$\sum_{n=1}^{\infty} \frac{1}{I(n)R(n)\varphi(n)} < e^2.$$  

He further proved that the arithmetic function

$$G(n) = \prod_{\nu=1}^{n} I(\nu)^{1/n}$$
satisfies the inequalities
\[ \frac{n}{e^2} < G(n) < n, \]
for each \( n \geq 1 \). Alkan and two of the authors [1] established an asymptotic formula for \( G(n) \) and proved that the sequence \( \{G(n)/n\}_{n \geq 1} \) is convergent. They further obtained results that show that \( I(n) \) is very regular on average. Further improvements have recently been obtained by Koinic and Kátai [7]. Asymptotic formulas for certain weighted real moments of \( R(n) \) were obtained in [9].

In the above more general setting, one realizes \( I(n) \) and \( R(n) \) as \( f_{A_1}(n) \) and \( f_{A_2}(n) \), respectively, with
\[
A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
and
\[
A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.
\]
Results on averages of \( f_A(n) \) have recently been established in [12]. That work generalizes \( I(n) \) and \( R(n) \) to a class of elements of \( \text{PSL}_2(\mathbb{Z}) \) and explores some of the properties of these maps.

For each given matrix \( A \) and a positive real number \( x \), we define the weighted average
\[
M_A(x) = \sum_{1 \leq n \leq x} \left( 1 - \frac{n}{x} \right) f_A(n).
\]
We also consider \( \lambda_A^+ \) and \( \lambda_A^- \), the positive and negative real eigenvalues of \( A \), respectively. Thus, \( \lambda_A^+ \) and \( \lambda_A^- \) are solutions of the quadratic equation
\[
\lambda^2 - \text{tr}(A) \lambda + \det(A) = 0,
\]
with
\[
\lambda_A^+ = \frac{a + d + \sqrt{(a + d)^2 + 4}}{2}
\]
and
\[
\lambda_A^- = \frac{a + d - \sqrt{(a + d)^2 + 4}}{2}. \tag{1}
\]
Furthermore, \( \lambda_A^+ \) and \( \lambda_A^- \) satisfy the inequalities \( \lambda_A^- < 0 < \lambda_A^+ \) and the identity \( \lambda_A^+ \lambda_A^- = -1 \).

In the present paper, for a large \( Q \) and a much larger \( x \), we consider the following subset of \( \text{PSL}_2(\mathbb{Z}) \):
\[
\mathcal{A}(Q, x) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 1 \leq a, b, c, d \leq Q, ad - bc = -1, \right. \\
\left. \left( \frac{\lambda_A^+}{Q}, Q\lambda_A^-, \frac{\log M_A(x)}{\log x} \right) \in \mathcal{S} \right\},
\]
where the surface $S$ is given by

$$S = \{(x, y, z) \in \mathbb{R}^3 : 1 < x, z < 2, xy = -1\}.$$  

(See Figure 1.)

The map

$$\Psi_{Q, x} : A(Q, x) \rightarrow S,$$

defined by

$$\Psi_{Q, x}(A) = \left(\frac{\lambda^+ A}{Q}, Q \lambda^- A, \frac{\log M_A(x)}{\log x}\right),$$

associates to each matrix $A \in A(Q, x)$ a unique point on $S$. In the first and second coordinates of such a point on $S$, the eigenvalues $\lambda^+_A$ and $\lambda^-_A$ of $A$ are normalized,
as $\lambda_A^+$ is divided by $Q$ and $\lambda_A^-$ is multiplied by $Q$. Furthermore, $\lambda_A^+$ is close to $a + d$, which can be $2Q$ at most. It follows that $\lambda_A^+/Q < 2$, with very few exceptions.

For the sake of simplicity, we restrict our attention to the case when $\lambda_A^+/Q$ is in the interval $(1, 2)$ and leave to the reader to make the adaptation to the case when $\lambda_A^+/Q$ is in the interval $(0, 1)$, as the two cases are similar.

In the third coordinate of such a point on $S$, we observe that for any $A$ with positive entries, $f_A(n) \geq 1$ for all $n$. It follows that $M_A(x) > x/2$. Hence,

$$\frac{\log M_A(x)}{\log x} > 1 - \frac{\log 2}{\log x}.$$ 

Finally, for simplicity’s sake, we consider only the case when $z$ is in the interval $(1, 2)$. In like manner, one can study the case when $z$ is in the interval $(2, \infty)$.

In the present paper, our purpose is to investigate the possible influence of the eigenvalues $\lambda_A^+$ and $\lambda_A^-$ of $A$ on the behavior of the associated arithmetic function $f_A(n)$. We seek to understand the joint distribution of $\lambda_A^+$, $\lambda_A^-$, and $(\log M_A(x))/\log x$, that is to say, the image of $\Psi_{Q,x}$ on $S$. More precisely, for a given point $(\alpha, -1/\alpha, \beta)$ on $S$ we consider, for each small $\delta > 0$, the neighborhood $V_{\alpha,\beta,\delta}$ of $(\alpha, -1/\alpha, \beta)$ in $S$ given by

$$V_{\alpha,\beta,\delta} = \{(x, y, z) \in S: |x - \alpha| < \delta, |z - \beta| < \delta\}.$$ 

We would like to estimate the number of matrices $A$ in $A(Q, x)$ for which $\Psi_{Q,x}(A)$ lies in $V_{\alpha,\beta,\delta}$. We expect the number of such matrices to grow like a constant times $\delta^2Q^2$ as $Q$ and $x$ tend to infinity, with $x$ much larger than $Q$, while $\delta > 0$ is kept fixed. This leads us to consider the limit of the ratio

$$\frac{\#\left\{\Psi_{Q,x}^{-1}(V_{\alpha,\beta,\delta})\right\}}{\delta^2Q^2} = \frac{\#\{A \in A(Q, x): \Psi_{Q,x}(A) \in V_{\alpha,\beta,\delta}\}}{\delta^2Q^2},$$ 

as $x$ approaches infinity and then $Q$ approaches infinity. Lastly, we take the limit of this expression as $\delta \to 0^+$.

Our main result can be summarized as follows.

**Theorem.** Fix a point $(\alpha, -1/\alpha, \beta) \in S$, where $\alpha$ and $\beta$ are real numbers such that $1 < \alpha, \beta < 2$. Then we have

$$\lim_{\delta \to 0} \lim_{Q \to \infty} \lim_{x \to \infty} \frac{\#\{A \in A(Q, x): \Psi_{Q,x}(A) \in V_{\alpha,\beta,\delta}\}}{\delta^2Q^2} = \begin{cases} \frac{24}{\pi^2} \left(\frac{\beta - \alpha}{\beta - 1}\right), & \text{if } \beta \geq \alpha; \\ 0, & \text{if } \beta < \alpha. \end{cases}$$

Thus, the images via $\Psi_{Q,x}$ of almost all matrices $A$ lie on the part of the surface $S$ where $z \geq x$, depicted in blue in Figure 1. If we fix two points $P_1 = (\alpha_1, -1/\alpha_1, \beta_1)$ and $P_2 = (\alpha_2, -1/\alpha_2, \beta_2)$ on that part of the surface $S$ and compare the local densities of the points in $\Psi_{Q,x}(A(Q, x))$ around $P_1$ and respectively $P_2$, as a direct consequence of our theorem we deduce the following corollary.
Corollary. Let $\alpha_j$ and $\beta_j$ be real numbers such that $1 < \alpha_j < \beta_j < 2$ for $j \in \{1, 2\}$. Then we have
\[
\lim_{\delta \to 0} \lim_{Q \to \infty} \lim_{x \to \infty} \# \{ A \in \mathcal{A}(Q,x) : \Psi_{Q,x}(A) \in \mathcal{V}_{\alpha_1,\beta_1,\delta} \} = (\beta_1 - \alpha_1)(\beta_2 - 1) - (\beta_1 - \alpha_1)(\beta_2 - \alpha_2)(\beta_1 - 1).
\]

2. Proof of the Theorem

We begin the proof by fixing an $\alpha$ and $\beta$ in the interval $(1,2)$ and a $\delta > 0$ small enough so that $\alpha$ and $\beta$ belong to the interval $(1+\delta, 2-\delta)$. We also consider the set of matrices
\[
\mathcal{D}_{\alpha,\beta,\delta,Q,x} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}(Q,x) : 1 \leq a, b, c, d \leq Q, \quad ad - bc = -1, \quad (\alpha - \delta)Q \leq a + d \leq (\alpha + \delta)Q, \quad (\beta - 1 - \delta)d < b < (\beta - 1 + \delta)d \right\}.
\]

The cardinality of $\mathcal{D}_{\alpha,\beta,\delta,Q,x}$ is given by
\[
\# \mathcal{D}_{\alpha,\beta,\delta,Q,x} = \sum_{1 \leq d \leq Q} \sum_{1 \leq c \leq d} \sum_{\gcd(c,d) = 1} \# \{(a, b) : 1 \leq a, b \leq d, \quad ad - bc = -1, \quad (\alpha - \delta)Q \leq a + d \leq (\alpha + \delta)Q, \quad (\beta - 1 - \delta)d < b < (\beta - 1 + \delta)d\}
\]
\[
= \sum_{1 \leq d \leq Q} \sum_{1 \leq c \leq d} \sum_{\gcd(c,d) = 1} 1,
\]

where $c$ is used to denote the unique multiplicative inverse of $c$ modulo $d$ in the interval $[1,d]$. The second step in (3) follows from the fact that the conditions $1 \leq b \leq d$ and $ad - bc = -1$ force $b$ to equal $c$. Hence, $a$ is uniquely determined and given by $a = (bc - 1)/d$. Furthermore, the contribution of the terms in (3) for which $d < (\alpha - \delta)Q/2$ is zero. Indeed, since $a \leq d$, we see that if $d < (\alpha - \delta)Q/2$, then $a + d < (\alpha - \delta)Q$.

Hence, setting $q = d$, $x = c$ and $y = \bar{c}$, we obtain $\# \mathcal{D}_{\alpha,\delta,Q}$ in the form
\[
\# \mathcal{D}_{\alpha,\beta,\delta,Q,x} = \sum_{(\alpha - \delta)Q/2 \leq q \leq Q} \# \{(x, y) : (x, y) \in \Omega_{\alpha,\beta,\delta,Q,q} \cap \mathbb{Z}^2 : xy = 1 \mod q\},
\]
where
\[
\Omega_{\alpha,\beta,\delta,Q,q} = \{(u, v) \in \mathbb{R}^2 : 1 \leq u, v \leq q, \quad (\alpha - \delta)qQ - q^2 \leq uv \leq (\alpha + \delta)qQ - q^2, \quad (\beta - 1 - \delta)q \leq v \leq (\beta - 1 + \delta)q\}.
\]
We estimate the summand in (4) by using a lemma due to Boca and Gologan [5].

**Lemma 1 (Lemma 2.3 from [5]).** Assume that \( q \geq 1 \) and \( h \) are two integers, that \( I \) and \( J \) are intervals of length less than \( q \), and that \( f: I \times J \to \mathbb{R} \) is a \( C^1 \) function. Then for any integer \( T > 1 \) and any \( \epsilon > 0 \), we have

\[
\sum_{a \in I, b \in J} f(a, b) = \frac{\phi(q)}{q^2} \int_{I} \int_{J} f(x, y) \, dx \, dy + \mathcal{E},
\]

with

\[
\mathcal{E} = O_\epsilon \left( T^2 \|f\|_\infty^{1/2+\epsilon} \gcd(h, q)^{1/2} + T \|\nabla f\|_\infty \|f\|_\infty^{3/2+\epsilon} \gcd(h, q)^{1/2} + \frac{\|\nabla f\|_\infty |I||J|}{T} \right),
\]

where \( \phi(q) \) is the Euler totient function, \( \|f\|_\infty \) and \( \|\nabla f\|_\infty \) denote the sup-norm of \( f \) and \( |\partial f/\partial x| + |\partial f/\partial y| \) on the region \( I \times J \), respectively.

We break the region \( \Omega_{\alpha, \beta, \delta, q} \) into squares of side length \( L = [Q^\eta] \) for some \( 0 < \eta < 1 \), and denote by \( I_j \) those squares lying entirely within \( \Omega_{\alpha, \beta, \delta, q} \) and \( B_i \) those squares which intersect both \( \Omega_{\alpha, \beta, \delta, q} \) and its complement in \( \mathbb{R}^2 \), where \( 1 \leq j \leq n \) and \( 1 \leq i \leq m \) for some natural numbers \( n \) and \( m \). We have

\[
\# \{ (u, v) \in \Omega_{\alpha, \beta, \delta, q} : ab \equiv 1 \text{ (mod } q) \} = \sum_{1 \leq j \leq n} \# \{ (u, v) \in I_j : ab \equiv 1 \text{ (mod } q) \} + \sum_{1 \leq i \leq m} \# \{ (u, v) \in B_i \cap \Omega_{\alpha, \beta, \delta, q} : ab \equiv 1 \text{ (mod } q) \}.
\]

By Lemma 1, each of the summands on the right-hand side above is equal to

\[
\frac{\phi(q)}{q^2} L^2 + O_\epsilon (q^{1/2+\epsilon}).
\]

If we take \( \Omega' \) to be the subset of \( \Omega_{\alpha, \beta, \delta, q} \) formed by removing from \( \Omega_{\alpha, \beta, \delta, q} \) an \( L\sqrt{2} \)-width neighborhood of the boundary of \( \Omega_{\alpha, \beta, \delta, q} \), then we find that \( \Omega' \subset \bigcup I_j \subset \Omega_{\alpha, \beta, \delta, q} \) and

\[
\text{Area} (\Omega_{\alpha, \beta, \delta, q}) - \text{Area}(\Omega') = O(qL).
\]

Hence,

\[
\text{Area} \left( \bigcup I_j \right) = \text{Area}(\Omega_{\alpha, \beta, \delta, q}) + O(QL).
\]

Since

\[
\text{Area} \left( \bigcup I_j \right) = \sum_{1 \leq j \leq n} \# \{ (u, v) \in I_j : ab \equiv 1 \text{ (mod } q) \} = n \frac{\phi(q)}{q^2} L^2 + O_\epsilon (nq^{1/2+\epsilon}),
\]
we have
\[ nL^2 = \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(QL), \]
and in particular
\[ n = O\left(\frac{Q^2}{L^2}\right). \]

Thus,
\[
\sum_{1 \leq j \leq n} \# \{(u, v) \in I_j : ab \equiv 1 \pmod{q}\} = n \frac{\phi(q)}{q^2} L^2 + O_\epsilon(nq^{1/2+\epsilon})
\]
\[
= \frac{\phi(q)}{q^2} (\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(QL))
\]
\[
+ O_\epsilon\left(\frac{Q^2}{L^2} q^{1/2+\epsilon}\right)
\]
\[
= \frac{\phi(q)}{q^2} \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(L)
\]
\[
+ O_\epsilon\left(\frac{Q^{5/2+\epsilon}}{L^2}\right). \]

Similarly, we find that \( m = O(Q/L) \) and
\[
0 \leq \sum_{1 \leq i \leq m} \# \{(u, v) \in B_i \cap \Omega_{\alpha,\beta,\delta,Q,q} : ab \equiv 1 \pmod{q}\}
\]
\[
\leq \sum_{1 \leq i \leq m} \# \{(u, v) \in B_i : ab \equiv 1 \pmod{q}\}
\]
\[
= m \frac{\phi(q)}{q^2} L^2 + O_\epsilon(mq^{1/2+\epsilon}) = O(L) + O_\epsilon\left(\frac{Q^{3/2+\epsilon}}{L}\right). \]

Taking \( \eta = 5/6 \), we have
\[
\# \{(u, v) \in \Omega_{\alpha,\beta,\delta,Q,q} : ab \equiv 1 \pmod{q}\} = \frac{\phi(q)}{q^2} \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O_\epsilon(Q^{5/6+\epsilon}). \]

Thus,
\[ \#D_{\alpha,\beta,\delta,Q,x} = M + E, \] (6)
where
\[
M = \sum_{(\alpha-\delta)Q/2 \leq q \leq Q} \frac{\phi(q)}{q^2} \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}), \] (7)
and
\[
E = \sum_{(\alpha-\delta)Q/2 \leq q \leq Q} E_{\alpha,\beta,\delta,Q,q} = O_\epsilon(Q^{11/6+\epsilon}). \] (8)
To examine the main term $M$ in (7), we recall from the definition of the set $\Omega_{\alpha,\beta,\delta,Q,q}$ in (5) that 

$$(\alpha - \delta)qQ - q^2 \leq uv \leq (\alpha + \delta)qQ - q^2.$$ 

We first note that when $\alpha > \beta$ and $\delta$ is small enough, all the areas $\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q})$ are zero for all values of $q$. Indeed, if $\alpha > \beta$ and $(u, v) \in \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q})$, then 

$$(\alpha - 1 - \delta)q^2 \leq (\alpha - \delta)qQ - q^2 \leq uv \leq (\beta - 1 + \delta)q^2.$$ 

This shows that for $\delta > 0$ small enough, all of the sets $\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q})$ are empty. In what follows we will restrict to the case $\alpha < \beta$. From the position of the hyperbolas $uv = (\alpha - \delta)qQ - q^2$ and $uv = (\alpha + \delta)qQ - q^2$, the horizontal lines $v = (p - 1 - \delta)q$ and $v = (p - 1 + \delta)q$, and their points of intersection with the boundary of the square $[1, q] \times [1, q]$, we find that 

$$\Omega_{\alpha,\beta,\delta,Q,q} = \mathcal{L} \cap ([1, q] \times [1, q]),$$ 

where $\mathcal{L}$ is the “parallelogram shaped” region that lies between the hyperbolas and horizontal lines.

It is easy to see that if $q < (\alpha - \delta)Q/(\beta + \delta)$, then $\mathcal{L}$ lies completely outside the square $[1, q] \times [1, q]$. Furthermore, one can verify that if $(\alpha - \delta)Q/(\alpha + \delta) \leq q \leq (\alpha + \delta)Q/(\beta - \delta)$, then $\mathcal{L}$ intersects the square $[1, q] \times [1, q]$ but does not lie entirely inside it. This forces $\mathcal{L}$ to lie close enough to the boundary of the square $[1, q] \times [1, q]$, so that the total contribution of these values of $q$ to the main term $M$ is negligible. Hence, we are left with the sum 

$$\sum_{(\alpha + \delta)Q/(\beta - \delta) \leq q \leq Q} \frac{\phi(q)}{q^2} \text{Area}(\mathcal{L}).$$  \hspace{1cm} (9)$$

Here, $\text{Area}(\mathcal{L})$ is asymptotic to the area of the parallelogram. That is, if $\delta$ is small enough, then we have 

$$\text{Area}(\mathcal{L}) \sim 2\delta q \left[ \frac{(\alpha + \delta)qQ - q^2}{(\beta - 1)q} - \frac{(\alpha - \delta)qQ - q^2}{(\beta - 1)q} \right] = 2\delta q \left( \frac{2\delta Q}{\beta - 1} \right) = 4\delta^2 qQ, \hspace{1cm} (10)$$

as $Q \to \infty$. Inserting (10) into (9), we obtain 

$$M \sim \frac{4\delta^2 Q}{\beta - 1} \sum_{(\alpha + \delta)Q/(\beta - \delta) \leq q \leq Q} \frac{\phi(q)}{q},$$ \hspace{1cm} (11)$$

We estimate the summation in (11) by employing the following result from [4].
Lemma 2 (Lemma 2.3 from [4]). Suppose that $a$ and $b$ are two real numbers such that $0 < a < b$, $q \in \mathbb{N}^*$ and $f$ is a piecewise $C^1$ function defined on $[a, b]$. Then we have

$$
\sum_{a < q \leq b} \frac{\phi(q)}{q} f(q) = \frac{1}{\zeta(2)} \int_a^b f(x) \, dx + O \left( \log b \left( \|f\|_\infty + \int_a^b |f'(x)| \, dx \right) \right).
$$

Applying Lemma 2, we get

$$
\sum_{(\alpha+\delta)Q/(\beta-\delta) \leq q \leq Q} \frac{\phi(q)}{q} = \frac{1}{\zeta(2)} \int_{(\alpha+\delta)Q/(\beta-\delta)}^Q dt + O(\log Q). \tag{12}
$$

Then inserting (12) into (11), we find that

$$
\frac{M}{\delta^2 Q^2} \rightarrow \frac{4}{(\beta-1)\zeta(2)} \left( 1 - \frac{\alpha}{\beta} \right), \tag{13}
$$

as $Q \to \infty$ first and then followed by $\delta \to 0$.

Next, we consider the set of matrices

$$
\mathcal{C}_{\alpha, \beta, \delta, Q, x} = \left\{ \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in \mathcal{A}(Q, x) : 1 \leq a, b, d \leq c \leq Q, \, ad - bc = -1, \right. \\
\left. (\alpha - \delta)Q \leq a + d \leq (\alpha + \delta)Q, \\
(\beta - 1 - \delta)c \leq a \leq (\beta - 1 + \delta)c \right\}.
$$

Estimating the cardinality of $\mathcal{C}_{\alpha, \beta, \delta, Q, x}$ in a similar fashion to that in (3), we write

$$
\#\mathcal{C}_{\alpha, \beta, \delta, Q, x} = \sum_{1 \leq c \leq Q} \sum_{\substack{1 \leq d \leq c \\ \gcd(c, d) = 1}} \frac{1}{(\alpha - \delta)Q \leq c - d + d \leq (\alpha + \delta)Q \\
(\beta - 1 - \delta)c \leq c - d \leq (\beta - 1 + \delta)c} \tag{14}
$$

The equality in (14) follows by noticing that the conditions $1 \leq a \leq c$ and $ad - bc = -1$ force $a$ to equal $c - d$, where $d$ is the multiplicative inverse of $d$ modulo $c$ in the interval $[1, c]$. Furthermore, let us note in (14) that the terms for which $c < (\alpha - \delta)Q/2$ have no contribution to the sum. Indeed, the inequality $(\alpha - \delta)Q \leq c - d + d$ implies $(\alpha - \delta)Q < 2q$. Hence, setting $q = c$, $x = d$ and $y = d$, we obtain

$$
\#\mathcal{C}_{\alpha, \beta, \delta, Q, x} \text{ in the form}
$$

$$
\#\mathcal{C}_{\alpha, \beta, \delta, Q, x} = \sum_{(\alpha - \delta)Q/2 \leq q \leq Q} \#\{ (x, y) \in \Gamma_{\alpha, \beta, \delta, Q, q} \cap \mathbb{Z}^2 : xy \equiv 1 \pmod{q} \}, \tag{15}
$$
where
\[ \Gamma_{\alpha,\beta,\delta,\mathbb{Q},q} = \{(u,v) \in \mathbb{R}^2 : 1 \leq u,v \leq q, \]
\[ (\alpha - \delta)Q - q \leq u - v \leq (\alpha + \delta)Q - q, \]
\[ (2 - \beta - \delta)q \leq v \leq (2 - \beta + \delta)q \}. \tag{16} \]

Applying Lemma 1 as before, we obtain
\[ \# \{(x,y) \in \Gamma_{\alpha,\beta,\delta,\mathbb{Q},q} \cap \mathbb{Z}^2 : xy \equiv 1 \pmod{q} \} = \frac{\phi(q)}{q^2} \text{Area}(\Gamma_{\alpha,\beta,\delta,\mathbb{Q},q}) \]
\[ + E_{\alpha,\beta,\delta,\mathbb{Q},q}, \tag{17} \]

where
\[ E_{\alpha,\beta,\delta,\mathbb{Q},q} = O_e(Q^{5/6+\epsilon}). \tag{18} \]

Then inserting (17) and (18) into (15), we get
\[ \#C_{\alpha,\beta,\delta,\mathbb{Q},x} = M' + E', \tag{19} \]

where
\[ M' = \sum_{(\alpha - \delta)Q/2 \leq q \leq Q} \frac{\phi(q)}{q^2} \text{Area}(\Gamma_{\alpha,\beta,\delta,\mathbb{Q},q}) \tag{20} \]

and
\[ E' = \sum_{(\alpha - \delta)Q/2 \leq q \leq Q} E'_{\alpha,\beta,\delta,\mathbb{Q},q} = O_e(Q^{11/6+\epsilon}). \tag{21} \]

From the definition of the set \( \Gamma_{\alpha,\beta,\delta,\mathbb{Q},q} \) in (16), we see that
\[ \Gamma_{\alpha,\beta,\delta,\mathbb{Q},q} = \mathcal{M} \cap ([1,q] \times [1,q]), \]

where \( \mathcal{M} \) is the parallelogram that lies between the slant lines \( v = u + q - (\alpha + \delta)Q \) and \( v = u + q - (\alpha - \delta)Q \) and the horizontal lines \( v = (\beta - \delta)q \) and \( v = (2 - \beta + \delta)q \).

First, we observe that if \( \alpha > \beta \), then for \( \delta \) small enough all parallelograms \( \mathcal{M} \) lie outside the square \([1,q] \times [1,q]\). In this situation, the sets \( \Gamma_{\alpha,\beta,\delta,\mathbb{Q},q} \) are empty. Hence, the main term \( M' \) is zero.

In what follows, we consider the case when \( \alpha < \beta \). If \( q < (\alpha - \delta)Q/(\beta + \delta) \), then the parallelograms \( \mathcal{M} \) still lie outside the square \([1,q] \times [1,q]\). Hence, we may restrict to the interval \([(\alpha - \delta)Q/(\beta + \delta), Q]\).

Next, if \( q \) belongs to the interval \([(\alpha - \delta)Q/(\beta + \delta), (\alpha + \delta)Q/(\beta - \delta)] \), then \( \mathcal{M} \) intersects the square \([1,q] \times [1,q]\) but is not entirely contained in it. This forces \( \mathcal{M} \) to lie close to the boundary of the square \([1,q] \times [1,q]\), so that all those values of \( q \) satisfying this property have negligible contribution to the main term \( M' \).

Hence, we may restrict the summation over \( q \) to the interval \([(\alpha + \delta)Q/(\beta - \delta), Q]\). For all such values of \( q \), we see that \( \mathcal{M} \) is entirely contained in the square \([1,q] \times [1,q]\).
and its area is equal to exactly $4\delta^2 qQ$. Hence, the main term in (20) is given by

$$M' = \sum_{(\alpha+\delta)Q/(\beta-\delta)\leq q\leq Q} \frac{\phi(q)}{q^2} \text{Area}(\Gamma_{\alpha,\beta,\delta,Q,q}) = 4\delta^2 Q \sum_{(\alpha+\delta)Q/(\beta-\delta)\leq q\leq Q} \frac{\phi(q)}{q}. \quad (22)$$

Using Lemma 2, we find that

$$\sum_{(\alpha+\delta)Q/(\beta-\delta)\leq q\leq Q} \frac{\phi(q)}{q} = \frac{Q}{2\zeta(2)} \left(1 - \frac{\alpha + \delta}{\beta - \delta}\right) + O(\log q). \quad (23)$$

Then inserting (23) into (22), we see that

$$\frac{M'}{\delta^2 Q^2} \to \frac{4}{\zeta(2)} \left(1 - \frac{\alpha + \delta}{\beta - \delta}\right), \quad (24)$$

as $Q \to \infty$ first and then followed by $\delta \to 0$. On combining the above estimates for $\#D_{\alpha,\beta,\delta,Q,x}$ and $\#C_{\alpha,\beta,\delta,Q,x}$ when $\beta$ is larger than $\alpha$ and recalling that both quantities are zero when $\beta$ is less than $\alpha$, we deduce that

$$\lim_{\delta \to 0} \lim_{Q \to \infty} \lim_{x \to \infty} \frac{\#D_{\alpha,\beta,\delta,Q,x} + \#C_{\alpha,\beta,\delta,Q,x}}{\delta^2 Q^2} = \begin{cases} 4 \left(1 - \frac{\alpha}{\beta}\right) \frac{1}{\zeta(2)} + 4 \left(1 - \frac{\alpha}{\beta}\right), & \text{if } \alpha \leq \beta; \\
0, & \text{if } \alpha < \beta; \\
\frac{4}{\zeta(2)} \left(\frac{\beta + \alpha}{\beta - 1}\right), & \text{if } \alpha \leq \beta; \\
0, & \text{if } \alpha > \beta. \end{cases} \quad (25)$$

We have the following result, which is essentially Theorem 1.1 from [12].

**Lemma 3.** Given a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of determinant $-1$ with $a, b, c, d \geq 1$, there are positive real-valued constants $K_A$ and $c'$ such that

$$M_A(x) = K_A x^{1+(a+b)/(c+d)} + O_A(x^{1/2+(a+b)/(c+d)} \exp\{-c'(\log x)^{3/5}(\log \log x)^{-1/5}\}).$$

For the sake of completeness, we outline a sketch of the proof of Lemma 3. Consider the Dirichlet series

$$F_A(s) = \sum_{n=1}^{\infty} \frac{f_A(n)}{n^s}.$$
One can show that \( F_A(s) \) converges in the half plane \( \Re s = \sigma > 1 + (a + b)/(c + d) \) and has an Euler product in that region. Write
\[
F_A(s) = \frac{\zeta(s - (a + b)/(c + d))}{\zeta(2s - 2(a + b)/(c + d))} T_A(s).
\]

Furthermore, one can show that \( \zeta(2s - 2(a + b)/(c + d))^{-1}T_A(s) \) is analytic on a larger half-plane \( \sigma > \sigma_0 \). Hence, \( F_A(s) \) is meromorphic there with a simple pole at \( s = 1 + (a + b)/(c + d) \).

Next, we utilize a variant of Perron’s formula and write
\[
\sum_{n \leq x} \left(1 - \frac{n}{x}\right) f_A(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s - (a + b)/(c + d))}{\zeta(2s - 2(a + b)/(c + d))} T_A(s) \frac{x^s}{s(s + 1)} ds,
\]
where \( 1 + (a + b)/(c + d) < c \leq 5/4 + (a + b)/(c + d) \). We need to apply the zero-free region for \( \zeta(s) \) due to Korobov [8] and Vinogradov [14] in the region
\[
\sigma \geq 1 - c_0 (\log t)^{-2/3} (\log \log t)^{-1/3}
\]
for \( t \geq t_0 \), in which
\[
\frac{1}{\zeta(\sigma)} = O((\log t)^{2/3} (\log \log t)^{1/3}).
\]

(See the end-of-chapter notes for Chapter 6 in Titchmarsh’s classical book [13]; see, also, Chapters 2 and 5 in Walfisz’s book [15].) We then fix \( 0 < U < T \leq x \), let \( \nu = 1/2 + (a + b)/(c + d) \) and
\[
\eta = \nu - c_0 (\log U)^{-2/3} (\log \log U)^{-1/3},
\]
and deform the path of integration into the union of the line segments
\[
\begin{align*}
\gamma_1, \gamma_9 : & s = c + it, \quad \text{if } |t| \geq T; \\
\gamma_2, \gamma_8 : & s = \sigma \pm iT, \quad \text{if } \nu \leq \sigma \leq c; \\
\gamma_3, \gamma_7 : & s = \nu + it, \quad \text{if } U \leq |t| \leq T; \\
\gamma_4, \gamma_6 : & s = \sigma \pm iU, \quad \text{if } \eta \leq \sigma \leq \nu; \\
\gamma_5 : & s = \eta + it, \quad \text{if } |t| \leq U.
\end{align*}
\]

Here, we note that the integrand is analytic on and within this modified contour. Hence, by the residue theorem
\[
M_A(x) = \frac{1}{(1 + (a + b)/(c + d))(2 + (a + b)/(c + d))\zeta(2)} T_A \left(1 + \frac{a + b}{c + d}\right) \\
\times x^{1+(a+b)/(c+d)} + \sum_{k=1}^{q} J_k,
\]
with the main term coming from the residue at the simple pole at \( s = 1 + (a + b)/(c + d) \). Note that we will take

\[
K_A = \frac{1}{(1 + (a + b)/(c + d))(2 + (a + b)/(c + d))\zeta(2)} T_A \left( 1 + \frac{a + b}{c + d} \right)
\]

in the statement of the lemma.

We estimate the integral along our modified contour and make use of the well-known bounds

\[
|\zeta(\sigma + it)| = \begin{cases} 
O(t^{(1-\sigma)/2}), & \text{if } 0 \leq \sigma \leq 1 \text{ and } |t| \geq 1; \\
O(\log t), & \text{if } 1 \leq \sigma \leq 2; \\
O(1), & \text{if } \sigma \geq 2.
\end{cases}
\]

(See Theorem 1.9 in Ivić’s classical book [6].) Upon collecting all estimates, we have the statement of the lemma.

Lemma 3 shows us that

\[
\frac{\log M_A(x)}{\log x} \sim 1 + \frac{a + b}{c + d},
\]

as \( x \to \infty \). Since

\[
a + b = a - \det(A) \quad \text{and} \quad \frac{b}{c} = \frac{b}{d} + \frac{\det(A)}{d(c + d)},
\]

when \( d > c \) we see that

\[
\left| \frac{\log M_A(x)}{\log x} - \frac{b}{d} \right| = O\left( \frac{1}{d^2} \right),
\]

as \( x \to \infty \). When \( c > d \), we have

\[
\left| \frac{\log M_A(x)}{\log x} - \frac{a}{c} \right| = O\left( \frac{1}{c^2} \right),
\]

as \( x \to \infty \).

We partition \( A(Q, x) \) into two subsets, according to whether \( 1 \leq \max(c, d) \leq \sqrt{Q} \) or \( \max(c, d) > \sqrt{Q} \). There are at most \( O(Q^{3/2}) \) matrices of the first type, and for the second type we have \( O(1/d^2) = O(1/Q) \) and \( O(1/c^2) = O(1/Q) \) when \( d > c \) and \( c > d \), respectively, as \( Q \to \infty \).

We note that the \( \delta \) in our definitions of \( D_{\alpha, \beta, \delta, Q, x} \) and \( C_{\alpha, \beta, \delta, Q, x} \) should be replaced by an expression of the form \( \delta + \delta_E(Q) \), where the function \( \delta_E(Q) = O(1/Q) \), but in what follows we let \( Q \) tend to infinity before letting \( \delta \) tend to zero, so in our case we may replace one by the other.

Since \( 1 + (a + b)/(c + d) < \beta + \delta < 2 \), we find that \( a < c \), and similarly \( b \leq d \). So the conditions \( a, b \leq d \) and \( a, b \leq c \) in \( D_{\alpha, \beta, \delta, Q, x} \) and \( C_{\alpha, \beta, \delta, Q, x} \) are satisfied. Thus,
\[
\lim_{x \to \infty} \left| \frac{\#D_{a,\beta,\delta;Q,x} + \#C_{a,\beta,\delta;Q,x}}{\delta^2 Q^2} - \frac{\#\{A \in \mathcal{A}(Q, x) : \Psi_{Q,x}(A) \in V_{a,\beta,\delta}\}}{\delta^2 Q^2} \right| = O\left(\frac{1}{\delta^2 \sqrt{Q}}\right)
\]
as \(Q \to \infty\). Upon combining this with (25), the theorem is proved.

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References


