MULTI-POLY-BERNOULLI-STAR NUMBERS AND FINITE
MULTIPLE ZETA-STAR VALUES

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Abstract
We define the multi-poly-Bernoulli-star numbers which generalize classical Bernoulli
numbers. We study the basic properties for these numbers and establish sum formulas
and a duality theorem, and discuss a connection to the finite multiple zeta-star
values. As an application, we present alternative proofs of some relations on the
finite multiple zeta-star values.

1. Introduction

For any multi-index \((k_1, \ldots, k_r)\) with \(k_i \in \mathbb{Z}\), we define two kinds of multi-poly-
Bernoulli-star numbers \(B_{n,*}^{(k_1, \ldots, k_r)}\), \(C_{n,*}^{(k_1, \ldots, k_r)}\) by the following generating series:

\[
\frac{Li_{k_1, \ldots, k_r}^*(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_{n,*}^{(k_1, \ldots, k_r)} \frac{t^n}{n!},
\]

\[
\frac{Li_{k_1, \ldots, k_r}^*(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} C_{n,*}^{(k_1, \ldots, k_r)} \frac{t^n}{n!},
\]

where \(Li_{k_1, \ldots, k_r}^*(z)\) is the non-strict multiple polylogarithm given by

\[
Li_{k_1, \ldots, k_r}^*(z) = \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{z^{m_1}}{m_1^{k_1} \cdots m_r^{k_r}}.
\]

When \(r = 1\), these numbers are poly-Bernoulli numbers studied in [1], [9]. Further,
when \(r = 1\) and \(k_1 = 1\), both numbers are classical Bernoulli numbers since \(Li_1^*(1 - e^{-t}) = t\). We note that \(B_{n,*}^{(1)} = C_{n,*}^{(1)} \ (n \neq 1)\) with \(B_{1,*}^{(1)} = 1/2\) and \(C_{1,*}^{(1)} = -1/2\).

We call \(k = k_1 + \cdots + k_r\) the weight of multi-index \((k_1, \ldots, k_r)\).

We set

\[
A := \prod_p \mathbb{Z}/p\mathbb{Z} = \{ (a_p)_p; a_p \in \mathbb{Z}/p\mathbb{Z} \}/ \sim,
\]
where \((a_p)_p \sim (b_p)_p\) is equivalent to the equalities \(a_p = b_p\) for all but finitely many primes \(p\). The finite multiple zeta-star values are defined by

\[
\zeta^*_\mathcal{A}(k_1, \ldots, k_r) := \left(H^*_p(k_1, \ldots, k_r) \mod p\right)_p \in \mathcal{A},
\]

where \(H^*_p(k_1, \ldots, k_r)\) is the non-strict multiple harmonic sum defined by

\[
H^*_n(k_1, \ldots, k_r) = \sum_{n-1 \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.
\]

For more details on the finite multiple zeta-star values, we refer the reader to [7], [10] and [12]. We use “star” to indicate that the inequalities in the sum are non-strict in contrast to the strict ones usually adopted in the references above. Each of these is expressed as a linear combination of the other.

This article is organized as follows. In §2, we give fundamental properties for the multi-poly-Bernoulli-star numbers. In §3, we describe the sum formula and the duality relation for the multi-poly-Bernoulli-star numbers. In §4, we study connections between the finite multiple zeta-star values and the multi-poly-Bernoulli-star numbers. As a result, we obtain alternative proofs of some relations for the finite multiple zeta-star values.

2. Basic Properties for the Multi-Poly-Bernoulli-Star Numbers

In this section, we introduce basic results for the multi-poly-Bernoulli-star numbers. We first give the recurrence relations for the multi-poly-Bernoulli-star numbers \(B^*_n(k_1, \ldots, k_r), C^*_n(k_1, \ldots, k_r)\). Before stating them, we provide the!following identity for the non-strict multiple polylogarithm, whose proof is straightforward and is omitted.

**Lemma 2.1.** For any multi-index \((k_1, \ldots, k_r)\) with \(k_i \in \mathbb{Z}\), we have

\[
\frac{d}{dt} \text{Li}^*_{k_1, \ldots, k_r}(t) = \begin{cases}
\frac{1}{t}\text{Li}^*_{k_1-1,k_2,\ldots,k_r}(t) & (k_1 \neq 1), \\
\frac{1}{t(1-t)}\text{Li}^*_{k_2-k_1,\ldots,k_r}(t) & (k_1 = 1, r \neq 1), \\
\frac{1}{1-t} & (k_1 = r = 1).
\end{cases}
\]

**Proposition 2.1.** For any multi-index \((k_1, \ldots, k_r)\), we have the following recursions:
(i) When \( k_1 \neq 1, \)

\[
B_{n,*}^{(k_1, \ldots, k_r)} = \frac{1}{n+1} \left( B_{n,*}^{(k_1-1, k_2, \ldots, k_r)} - \sum_{j=1}^{n-1} \binom{n}{j} B_{j,*}^{(k_1, \ldots, k_r)} \right),
\]

\[
C_{n,*}^{(k_1, \ldots, k_r)} = \frac{1}{n+1} \left( C_{n,*}^{(k_1-1, k_2, \ldots, k_r)} - \sum_{j=0}^{n-1} \binom{n+1}{j} C_{j,*}^{(k_1, \ldots, k_r)} \right).
\]

(ii) When \( k_1 = 1, \)

\[
B_{n,*}^{(1, k_2, \ldots, k_r)} = \frac{1}{n+1} \left( \sum_{j=0}^{n} \binom{n}{j} B_{j,*}^{(k_2, \ldots, k_r)} - \sum_{j=1}^{n-1} \binom{n}{j-1} B_{j,*}^{(1, k_2, \ldots, k_r)} \right),
\]

\[
C_{n,*}^{(1, k_2, \ldots, k_r)} = \frac{1}{n+1} \left( C_{n,*}^{(k_2, \ldots, k_r)} - \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j-1} C_{j,*}^{(1, k_2, \ldots, k_r)} \right),
\]

where an empty sum is understood to be 0.

Proof. We prove the relations for \( B_{n,*}^{(k_1, \ldots, k_r)} \): those for \( C_{n,*}^{(k_1, \ldots, k_r)} \) are similar.

\[
\ln_{k_1, \ldots, k_r} (1 - e^{-t}) = (1 - e^{-t}) \sum_{n=0}^{\infty} B_{n,*}^{(k_1, \ldots, k_r)} \frac{t^n}{n!}.
\]  

(1)

We differentiate both sides of (1): When \( k_1 \neq 1, \) we obtain

\[
\text{(LHS)} = \frac{e^{-t}}{1 - e^{-t}} \ln_{k_1-1, k_2, \ldots, k_r} (1 - e^{-t}),
\]

\[
\text{(RHS)} = e^{-t} \sum_{n=0}^{\infty} B_{n,*}^{(k_1, \ldots, k_r)} \frac{t^n}{n!} + (1 - e^{-t}) \sum_{n=0}^{\infty} B_{n+1,*}^{(k_1, \ldots, k_r)} \frac{t^n}{n!}.
\]

So we have

\[
\sum_{n=0}^{\infty} \left( B_{n,*}^{(k_1-1, k_2, \ldots, k_r)} - B_{n,*}^{(k_1, \ldots, k_r)} \right) \frac{t^n}{n!} = (e^t - 1) \sum_{n=0}^{\infty} B_{n+1,*}^{(k_1, \ldots, k_r)} \frac{t^n}{n!}
\]

\[
= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n}{j} B_{j+1,*}^{(k_1, \ldots, k_r)} \frac{t^n}{n!}.
\]

Comparing the coefficients of \( t^n/n! \) on both sides, we obtain (i). When \( k_1 = 1, \)

\[
\text{(LHS)} = \frac{1}{1 - e^{-t}} \ln_{k_2, \ldots, k_r} (1 - e^{-t}),
\]

\[
\text{(RHS)} = \sum_{n=0}^{\infty} B_{n+1,*}^{(1, k_2, \ldots, k_r)} \frac{t^n}{n!} + e^{-t} \sum_{n=0}^{\infty} \left( B_{n,*}^{(1, k_2, \ldots, k_r)} - B_{n+1,*}^{(1, k_2, \ldots, k_r)} \right) \frac{t^n}{n!}.
\]
So we have
\[
\sum_{n=0}^{\infty} \left( B_{n,^*}^{(1,k_2,\ldots,k_r)} - B_{n+1,^*}^{(1,k_2,\ldots,k_r)} \right) \frac{t^n}{n!} = e^t \sum_{n=0}^{\infty} \left( B_{n,^*}^{(k_2,\ldots,k_r)} - B_{n+1,^*}^{(1,k_2,\ldots,k_r)} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \left( B_{j,^*}^{(k_2,\ldots,k_r)} - B_{j+1,^*}^{(1,k_2,\ldots,k_r)} \right) \frac{t^n}{n!}
\]
and by this we obtain (ii).

We proceed to describe explicit formulas for the multi-poly-Bernoulli-star numbers in terms of the Stirling numbers of the second kind. We recall that the Stirling numbers of the second kind are the integers \( \binom{n}{m} \) for all integers \( m,n \) satisfying the following recursions and the initial values:

\[
\begin{align*}
\binom{n+1}{m} &= \binom{n}{m-1} + m \binom{n}{m} \quad (\forall n, m \in \mathbb{Z}), \\
\binom{0}{0} &= 1, \quad \binom{n}{0} = 0, \quad \binom{0}{m} = 0 \quad (n, m \neq 0).
\end{align*}
\]

**Proposition 2.2.** For any multi-index \( (k_1, \ldots, k_r), k_i \in \mathbb{Z} \), we have

\[
B_{n,^*}^{(k_1,\ldots,k_r)} = \sum_{n+1 \geq m_1 \geq \cdots \geq m_r \geq 1} (-1)^{m_1+n-1} (m_1-1)! \binom{n}{m_1-1} \frac{m_1^{k_1} \cdots m_r^{k_r}}{m_1^{k_1} \cdots m_r^{k_r}}
\]

and

\[
C_{n,^*}^{(k_1,\ldots,k_r)} = \sum_{n+1 \geq m_1 \geq \cdots \geq m_r \geq 1} (-1)^{m_1+n-1} (m_1-1)! \binom{n+1}{m_1} \frac{m_1^{k_1} \cdots m_r^{k_r}}{m_1^{k_1} \cdots m_r^{k_r}}.
\]

**Proof.** Using the following identity (cf. [3, eqn. 7.49])

\[
(e^t - 1)^m = m! \sum_{j=m}^{\infty} \binom{j}{m} \frac{t^j}{j!} \quad (m \geq 0),
\]

we have

\[
\sum_{n=0}^{\infty} B_{n,^*}^{(k_1,\ldots,k_r)} \frac{t^n}{n!} = \frac{\log(1-e^{-t})}{1-e^{-t}} = \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{(1-e^{-t})^{m_1-1}}{m_1^{k_1} \cdots m_r^{k_r}} = \sum_{m_1 \geq \cdots \geq m_r \geq 1} \sum_{j=m_1-1}^{\infty} (-1)^{m_1+j-1} (m_1-1)! \binom{j}{m_1-1} \frac{t^j}{j!}
\]
\[
\sum_{j=0}^{\infty} \sum_{j+1 \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{(-1)^{m_1+j-1}(m_1-1)!}{m_1^{k_1} \cdots m_r^{k_r}} \binom{j}{m_1-1} t^j.
\]
Thus comparing the coefficients of \(t^n/n!\) on both sides, we obtain the explicit formula for \(B_{n,*}^{(k_1,\ldots,k_r)}\).

The explicit formula for \(C_{n,*}^{(k_1,\ldots,k_r)}\) is obtained similarly by using the identity
\[
\frac{e^{-t}(1-e^{-t})^{m-1}}{(m-1)!} = \sum_{j=m-1}^{\infty} \frac{(-1)^{j+m+1}}{m!} \binom{j}{m} t^j,
\]
which follows from (2) by differentiation.

We finish this section by giving some simple relations among the multi-poly-Bernoulli-star numbers.

**Proposition 2.3.** For any multi-index \((k_1,\ldots,k_r)\) with \(k_i \in \mathbb{Z}\), we have
\[
B_{n,*}^{(k_1,\ldots,k_r)} = \sum_{j=0}^{n} \binom{n}{j} C_{j,*}^{(k_1,\ldots,k_r)}
\]
and
\[
C_{n,*}^{(k_1,\ldots,k_r)} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_{j,*}^{(k_1,\ldots,k_r)}.
\]

**Proof.** The generating functions of \(B_{n,*}^{(k_1,\ldots,k_r)}\), \(C_{n,*}^{(k_1,\ldots,k_r)}\) differ by the factor \(e^t\), and the above identities follow immediately.

**Proposition 2.4.** For any multi-index \((k_1,\ldots,k_r)\), we have
\[
C_{n,*}^{(k_1,\ldots,k_r)} = B_{n,*}^{(k_1,\ldots,k_r)} - C_{n-1,*}^{(k_1-1,k_2,\ldots,k_r)}.
\]

**Proof.** By the explicit formula for \(C_{n,*}^{(k_1,\ldots,k_r)}\), we obtain
\[
C_{n,*}^{(k_1,\ldots,k_r)} = \sum_{n+1 \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{(-1)^{m_1+n-1}(m_1-1)!}{m_1^{k_1} \cdots m_r^{k_r}} \binom{n+1}{m_1} m_1
\]
\[
= \sum_{n+1 \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{(-1)^{m_1+n-1}(m_1-1)!}{m_1^{k_1} \cdots m_r^{k_r}} \binom{n}{m_1-1} m_1
\]
\[
+ \sum_{n+1 \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{(-1)^{m_1+n-1}(m_1-1)!}{m_1^{k_1-1}m_2^{k_2} \cdots m_r^{k_r}} \binom{n}{m_1}
\]
\[
= B_{n,*}^{(k_1,\ldots,k_r)} - C_{n-1,*}^{(k_1-1,k_2,\ldots,k_r)}.
\]
The second equality above is by the recursion for the Stirling numbers of the second kind.
3. Main Results

We give the following sum formulas for multi-poly-Bernoulli-star numbers.

**Theorem 3.1.** We have

\[
\sum_{k_1 + \cdots + k_r = k \atop 1 \leq r \leq k, k_i \geq 1} (-1)^r B_{n,*}^{(k_1, \ldots, k_r)} = \frac{(-1)^k}{k} \binom{n}{k-1} B_{n-k+1,*}^{(1)} \tag{3}
\]

and

\[
\sum_{k_1 + \cdots + k_r = k \atop 1 \leq r \leq k, k_i \geq 1} (-1)^r C_{n,*}^{(k_1, \ldots, k_r)} = \frac{(-1)^k}{k} \binom{n}{k-1} C_{n-k+1,*}^{(1)}. \tag{4}
\]

**Proof.** We multiply both sides of (3) by $t^n/n!$ and sum on $n$. Hence we have

\[
\text{(LHS)} = \sum_{n=0}^{\infty} \sum_{k_1 + \cdots + k_r = k \atop 1 \leq r \leq k, k_i \geq 1} (-1)^r B_{n,*}^{(k_1, \ldots, k_r)} \frac{t^n}{n!}.
\]

\[
= \sum_{k_1 + \cdots + k_r = k \atop 1 \leq r \leq k, k_i \geq 1} (-1)^r \frac{t^n}{n!} \frac{Li_{k_1, \ldots, k_r}^* (1 - e^{-t})}{1 - e^{-t}}.
\]

\[
\text{(RHS)} = \frac{(-1)^k}{k} \sum_{n=0}^{\infty} \binom{n}{k-1} B_{n-k+1,*}^{(1)} \frac{t^n}{n!}
\]

\[
= \frac{(-1)^k}{k} \sum_{n=0}^{\infty} B_{n-k+1,*}^{(1)} \frac{t^n}{(n-k+1)!}.
\]

\[
= \frac{(-1)^k}{k} \sum_{n=0}^{\infty} B_{n,*}^{(1)} \frac{t^{n+k-1}}{n!}.
\]

\[
= \frac{(-1)^k}{k} \frac{t^k}{1 - e^{-t}}.
\]

Since both sides have the same denominator, it suffices to prove the following identity:

\[
\sum_{k_1 + \cdots + k_r = k \atop 1 \leq r \leq k, k_i \geq 1} (-1)^r Li_{k_1, \ldots, k_r}^* (1 - e^{-t}) = \frac{(-1)^k}{k!} t^k. \tag{5}
\]

This equality is proved by induction on the weight. When $k = 1$, the left-hand side is $-Li_1^* (1 - e^{-t}) = -t$ and is equal to the right-hand side. Next we assume the
identity holds when the weight is $k$. Then by differentiating the left-hand side of the identity of weight $k + 1$, we obtain

$$
\frac{d}{dt} \left( \sum_{k_1 + \cdots + k_r = k+1 \atop 1 \leq r \leq k; k_i \geq 1} (-1)^r L^*_i \right) (1 - e^{-t})
$$

\[ = \sum_{k_1 + \cdots + k_r = k \atop 1 \leq r \leq k; k_i \geq 1} (-1)^r \left( \frac{e^{-t}}{1 - e^{-t}} - \frac{1}{1 - e^{-t}} \right) L^*_i (1 - e^{-t}) \]

\[ = \sum_{k_1 + \cdots + k_r = k \atop 1 \leq r \leq k; k_i \geq 1} (-1)^{r+1} L^*_i (1 - e^{-t}) \]

\[ = (-1)^{k+1} \frac{k^k}{k!} t^k. \]

We used the induction hypothesis in the last equality. Therefore we have

$$
\sum_{k_1 + \cdots + k_r = k+1 \atop 1 \leq r \leq k; k_i \geq 1} (-1)^r L^*_i (1 - e^{-t}) = \frac{(-1)^{k+1}}{(k+1)!} t^{k+1} + C
$$

with some constant $C$, which we find is 0 by putting $t = 0$. The equation (4) follows from (5) since the generating function of $C_{n,*}$ differs from that of $B_{n,*}$ only by a factor $e^{-t}$.

Next we describe the duality relation for the multi-poly-Bernoulli-star numbers. We recall the duality operation of Hoffman [6, p. 65]. We define a function $S$ from the set of multi-indices $(k_1, \ldots, k_r)$ with $k_i \geq 1$ and weight $k$ to the power set of \{1, 2, \ldots, k - 1\} by

$$
S((k_1, \ldots, k_r)) = \{k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_r - 1\}.
$$

Obviously, the map $S$ is a one-to-one correspondence. Then $(k'_1, \ldots, k'_r)$ is said to be the dual index for $(k_1, \ldots, k_r)$ in Hoffman’s sense when

$$
(k'_1, \ldots, k'_r) = S^{-1}(\{1, 2, \ldots, k - 1\} - S((k_1, \ldots, k_r))).
$$

It is easy to see that Hoffman’s duality operation is an involution. Note that $k_1 > 1$ if and only if $k'_1 = 1$.

**Theorem 3.2.** For any multi-index $(k_1, \ldots, k_r)$ with $k_i \geq 1 (1 \leq i \leq r)$, we have

$$
C^{(k_1, \ldots, k_r)}_{n,*} = (-1)^n B^{(k'_1, \ldots, k'_r)}_{n,*},
$$

where $(k'_1, \ldots, k'_r)$ is the dual index of $(k_1, \ldots, k_r)$ in Hoffman’s sense.
Proof. As in the previous proof, consider the generating functions of both sides:

\[(\text{LHS}) = \sum_{n=0}^{\infty} C_{n,*}^{(k_1,\ldots,k_r)} \frac{t^n}{n!} = \frac{Li_{k_1,\ldots,k_r}^*(1-e^{-t})}{e^t-1},\]

\[(\text{RHS}) = \sum_{n=0}^{\infty} (-1)^n B_{n,*}^{(k_1',\ldots,k_r')} \frac{t^n}{n!} = \frac{Li_{k_1',\ldots,k_r'}^*(1-e^t)}{1-e^t}.\]

Hence we have to show the following identity:

\[Li_{k_1,\ldots,k_r}^*(1-e^{-t}) + Li_{k_1',\ldots,k_r'}^*(1-e^t) = 0.\]

This identity also follows from induction on the weight. First, it is trivial in the case \(k = 1\). Thus, we assume the above identity holds when the weight is \(k\). Since \(k_1 = 1\) is equivalent to \(k_1' \neq 1\), we may assume \(k_1 = 1\) by the symmetry of the identity. Then when the weight is \(k + 1\), the derivative of the left-hand side yields

\[
\frac{d}{dt} \left( Li_{k_1,\ldots,k_r}^*(1-e^{-t}) + Li_{k_1',\ldots,k_r'}^*(1-e^t) \right) \\
= \frac{1}{1-e^{-t}} Li_{k_1,\ldots,k_r}^*(1-e^{-t}) + \frac{-e^t}{1-e^t} Li_{k_1',\ldots,k_r'}^*(1-e^t) \\
= \frac{1}{1-e^{-t}} \left( Li_{k_1,\ldots,k_r}^*(1-e^{-t}) + Li_{k_1',\ldots,k_r'}^*(1-e^t) \right) \\
= 0.
\]

Therefore we obtain

\[Li_{k_1,\ldots,k_r}^*(1-e^{-t}) + Li_{k_1',\ldots,k_r'}^*(1-e^t) = C\]

with some constant \(C\), and by putting \(t = 0\), we conclude \(C = 0\). \(\square\)

4. Connection to the Finite Multiple Zeta-Star Values

In this section, we give alternative proofs for some relations of the finite multiple zeta-star values using the multi-poly-Bernoulli-star numbers. The following congruence is the “star-version” of the congruence given in [8, Theorem.8], and is proved in exactly the same manner:

\[H_p^*(k_1,\ldots,k_r) \equiv -C_{p-2,*}^{(k_1-1,k_2,\ldots,k_r)} \mod p.\]

Thus we find

\[\zeta_{A}(k_1,\ldots,k_r) = \left(-C_{p-2,*}^{(k_1-1,k_2,\ldots,k_r)} \mod p\right)_p, \tag{6}\]
Corollary 4.1 (M. Hoffman [7]). For any multi-index \((k_1, \ldots, k_r)\) with \(k_i \geq 1\) for \(1 \leq i \leq r\), let \((k'_1, \ldots, k'_r)\) be the dual index for \((k_1, \ldots, k_r)\) in Hoffman’s sense. Then we have
\[
\zeta_A^*(k_1, \ldots, k_r) = -\zeta_A^*(k'_1, \ldots, k'_r).
\]

**Proof.** It is sufficient to prove the case \(k_1 = 1\). By (6) and the duality relation for the multi-poly-Bernoulli-star numbers, we obtain
\[
\text{LHS} = \left( -C_{p-2,*}^{(0,k_2,\ldots,k_r)} \mod p \right)_p,
\]
\[
\text{RHS} = \left( C_{p-2,*}^{(k'_1-1,k'_2,\ldots,k'_r)} \mod p \right)_p
\]
\[
= \left( (-1)^p B_{p-2,*}^{(k_2,\ldots,k_r)} \mod p \right)_p.
\]

Hence we complete the proof if we prove \(C_{n,*}^{(0,k_2,\ldots,k_r)} = B_{n,*}^{(k_2,\ldots,k_r)}\) for all \(n\). We consider the generating functions of these numbers:
\[
\sum_{n=0}^{\infty} C_{n,*}^{(0,k_2,\ldots,k_r)} \frac{t^n}{n!} = \frac{Li_{0,k_2,\ldots,k_r}^*(1 - e^{-t})}{e^t - 1}
\]
\[
= \frac{1}{e^t - 1} \sum_{m_2 \geq \cdots \geq m_r \geq 1} \frac{1}{m_2^{k_2} \cdots m_r^{k_r}} \sum_{m_1 = m_2}^{\infty} (1 - e^{-t})^{m_1}
\]
\[
= \frac{1}{e^{-t}(e^t - 1)} Li_{k_2,\ldots,k_r}^*(1 - e^{-t})
\]
\[
= \sum_{n=0}^{\infty} B_{n,*}^{(k_2,\ldots,k_r)} \frac{t^n}{n!}.
\]

From this we have
\[
-C_{p-2,*}^{(0,k_2,\ldots,k_r)} = (-1)^p B_{p-2,*}^{(k_2,\ldots,k_r)}
\]
for any odd prime \(p\).

The following corollary is a weaker version of the sum formula for the finite multiple zeta-star values proved in [11].

**Corollary 4.2.** We have
\[
\sum_{\substack{k_1 + \cdots + k_r = k \\ r \geq 1, k_1 \geq 2, k_i \geq 1}} (-1)^r \zeta_A^*(k_1, \ldots, k_r) = (B_{p-k} \mod p)_p,
\]
where \(B_n\) is the classical Bernoulli numbers.
Proof. Equations (4) and (6) yield
\[
\sum_{k_1 + \cdots + k_r = k+1 \atop r \geq 1, k_1 \geq 2, k_i \geq 1} (-1)^{r+1} \zeta_A^{(k)}(k_1, \ldots, k_r) = \left( \frac{(-1)^k}{k} \left( \frac{p-2}{k-1} \right) c^{(1)}_{p-k-1,*} \mod p \right)_p
\]
\[
= \left( -c^{(1)}_{p-k-1,*} \mod p \right)_p.
\]
So replacing \( k \) by \( k-1 \), we obtain the desired identity. \( \square \)

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References