RAMSEY FUNCTIONS FOR GENERALIZED PROGRESSIONS

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Abstract
Given positive integers $m$ and $k$, a $k$-term semi-progression of scope $m$ is a sequence $x_1, x_2, \ldots, x_k$ such that $x_{j+1} - x_j \in \{d, 2d, \ldots, md\}$, for $1 \leq j \leq k-1$, for some positive integer $d$. Thus an arithmetic progression is a semi-progression of scope 1. Let $S_m(k)$ denote the least integer for which every 2-coloring of $\{1, 2, \ldots, S_m(k)\}$ yields a monochromatic $k$-term semi-progression of scope $m$. We obtain an exponential lower bound on $S_m(k)$ for all $m = O(1)$. Our approach also yields a marginal improvement on the best known lower bound for the analogous Ramsey function for quasi-progressions, which are sequences whose successive differences lie in a small interval.

1. Introduction
In 1927, B.L. van der Waerden [6] proved that given positive integers $r$ and $k$, there exists an integer $W(r, k)$ such that any $r$-coloring of $\{1, 2, \ldots, W(r, k)\}$ yields a monochromatic $k$-term arithmetic progression. Even after nearly 90 years, the gap between the lower and upper bounds on $W(r, k)$ remains enormous, with the best known lower bound of the order of $r^k$, whereas the best known upper bound is a five-times iterated tower of exponents (see [1]). Analogues of the Van der Waerden threshold $W(r, k)$ have been studied for many variants of arithmetic progressions, including semi-progressions and quasi-progressions (see [4]).

Let $m$ and $k$ be positive integers. A $k$-term semi-progression of scope $m$ is a sequence $x_1, x_2, \ldots, x_k$ such that for some positive integer $d$, we have $x_{j+1} - x_j \in \{d, 2d, \ldots, md\}$. The integer $d$ is called the low-difference of the semi-progression.
We define $S_m(k)$ as the least integer for which any 2-coloring of $\{1, 2, \ldots, S_m(k)\}$ yields a monochromatic $k$-term semi-progression of scope $m$. Note that $S_m(k) \leq W(k)$ with equality if $m = 1$.

2. An Exponential Lower Bound for $S_m(k)$

Landman [3] showed that $S_m(k) \geq (2k^2/m)(1 + o(1))$. We improve this to an exponential lower bound for all $m = O(1)$.

**Theorem 1** Let $k \geq 3, m = O(1)$ and $\alpha = \sqrt{2^m/(2^m - 1)}$. Then $S_m(k) > \alpha^k$.

**Proof.** Let $f(N, k, m)$ denote the number of 2-colorings of $[1, N]$ with a monochromatic $k$-term semi-progression of scope $m$. (In the remainder of the proof, we only consider $k$-term semi-progressions of scope $m$.) Note that $S_m(k)$ is the least integer $N$ such that $f(N, k, m) = 2^N$. We derive an upper bound on $f(N, k, m)$ as follows.

Given a semi-progression $P = (a_1, a_2, \ldots, a_k)$ of low-difference $d$, we define the conjugate vector of $P$ as $(u_1, u_2, \ldots, u_{k-1})$ where $u_i = (a_i + 1 - a_i - d)/d$. Likewise, the frequency vector of $P$ is defined as $v = (v_0, v_1, \ldots, v_{m-1})$ where $v_j$ is the number of times $j$ occurs in the conjugate vector of $P$. Note that $\sum_{j=0}^{m-1} v_j = k - 1$. Finally, the weight of a frequency vector $v$, denoted $w(v)$, is defined as $\sum_{j=0}^{m-1} jv_j$.

Given a coloring $\chi$, we define the $(a, d)$-primary semi-progression of $\chi$ as the semi-progression $P$ whose conjugate vector is lexicographically least among the conjugate vectors of all semi-progressions (with first term $a$ and low-difference $d$) that are monochromatic under $\chi$. Let $P = (a_1, a_2, \ldots, a_k)$ be a semi-progression with first term $a_1 = a$ and low-difference $d$. We will give an upper bound for the number of colorings $\chi$ such that $P$ is the $(a, d)$-primary semi-progression of $\chi$.

Since $P$ is monochromatic, all elements of $P$ have the same color under $\chi$. Furthermore, if $(v_0, v_1, \ldots, v_{m-1})$ is the frequency vector of $P$, it follows from the fact that $P$ is the $(a, d)$-primary semi-progression of $\chi$ that $w(v)$ elements in the arithmetic progression $\{a, a+d, \ldots, a+m(k-1)d\}$ must be of the color different from the color of the elements of $P$. For example, let $a = 17, d = 5, m = 3, k = 6$ and $P = \{17, 32, 42, 47, 62, 72\}$ with conjugate vector $(2, 1, 0, 2, 1)$. If the two colors are red and blue, and the elements of $P$ are all red, then $22, 27, 37, 52, 57$ and $67$ must all be blue. Indeed, if $57$ is red, then the semi-progression $P' = \{17, 32, 42, 47, 57, 62\}$ would have a lexicographically lower conjugate vector $(2, 1, 0, 1, 0)$. Thus there are at most $2^{N-11}$ colorings of $[1, N]$ whose $(a, d)$-primary semi-progression is $P$.

Note that there are at most $N^2/(k-1)$ choices for the pair $(a, d)$. We say that two progressions $P_1$ and $P_2$ with the same $a$ and $d$ are equivalent if they have the
same frequency vector. Note that for any \(a\) and \(d\), there are at most

\[
M(\mathbf{v}) = \frac{(v_0 + v_1 + \cdots + v_{m-1})!}{v_0!v_1! \cdots v_{m-1}!}
\]

semi-progressions with frequency vector \((v_0, v_1, \ldots, v_{m-1})\). Adding over all the equivalence classes of semi-progressions, we obtain

\[
f(N, k, m) \leq \frac{N^22^{N-k+1}}{k-1} \sum_{v_0, v_1, \ldots, v_{m-1} \geq 0} M(\mathbf{v})2^{-w(\mathbf{v})}
\]

It follows from the multinomial theorem that

\[
f(N, k, m) \leq \frac{N^22^N}{k-1} \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^m}\right)^{k-1}
\]

Thus \(f(N, k, m) < 2^N\) for \(N = \alpha_m^k\) where \(\alpha_m = \sqrt{2^m/(2^m - 1)}\), completing the proof. \(\square\)

3. Exponential Lower Bounds for \(Q_n(r, k)\)

We now apply the same technique to quasi-progressions. A \(k\)-term quasi-progression of low difference \(d\) and diameter \(n\) is a sequence \((a_1, a_2, \ldots, a_k)\) such that \(d \leq a_{j+1} - a_j \leq d + n, 1 \leq j \leq k - 1\). Let \(Q_n(r, k)\) denote the least positive integer such that any \(r\)-coloring of \([1, 2, \ldots, Q_n(r, k)]\) yields a monochromatic \(k\)-term quasi-progression of diameter \(n\). It is known (see [5]) that \(Q_1(2, k) > \beta_k\) where \(\beta = 1.08226\ldots\). Indeed, \(\beta^4\) can be expressed in terms of two algebraic numbers of degrees 2 and 3, respectively, and is the smallest positive real root of the equation

\[
y^6 + 8y^5 - 112y^4 - 128y^3 + 1792y^2 + 1024y - 4096 = 0.
\]

It is also known that \(Q_1(2, k) = O(k^2)\) for \(n > k/2\) (see [2]). We apply the techniques of the previous section to obtain lower bounds on \(Q_n(r, k)\). Let \(g(r, N, k, n)\) denote the number of \(r\)-colorings of \([1, N]\) with a monochromatic \(k\)-term semi-progression of diameter \(n\). Note that \(Q_n(r, k)\) is the least positive integer \(N\) such that \(g(r, N, k, n) = r^N\). We first discuss the simplest non-trivial case, namely \(r = 2\) and \(n = 1\).

**Theorem 2** Let \(k \geq 3\). Then \(Q_1(2, k) > \beta_{2,1}^k\), where \(\beta_{2,1} = \sqrt{4 - 2\sqrt{2}} = 1.08239\ldots\).

**Proof.** We define the conjugate vector of a quasi-progression \(Q = \{a_1, a_2, \ldots, a_k\}\) of low-difference \(d\) as \((u_1, u_2, \ldots, u_{k-1})\) where \(u_i = a_{i+1} - a_i - d\). Given a coloring \(\chi\), we define the \((a, d)\)-primary quasi-progression of \(\chi\) as the quasi-progression \(Q\) whose conjugate vector is lexicographically least among the conjugate vectors of all quasi-progressions (with first term \(a\) and low-difference \(d\)) that are monochromatic.
under $\chi$. Let $Q = \{a_1, a_2, \ldots, a_k\}$ be a quasi-progression with first term $a_1 = a$ and low-difference $d$. We give an upper bound for the number of colorings $\chi$ such that $Q$ is the $(a, d)$-primary quasi-progression of $\chi$.

Since $Q$ is monochromatic, all elements of $Q$ have the same color under $\chi$, say red. Let $(u_1, u_2, \ldots, u_{k-1})$ be the conjugate vector of $Q$. Observe that if $u_j = 1$ and $u_{j+1} = 0$ for some $j$, so that $a_j, a_j + d + 1$ and $a_j + 2d + 1$ are elements of $Q$, and therefore red, it follows that the color of $a_j + d$ is different from red (say blue), as $(P \cup \{a_j + d\}) \setminus \{a_j + d + 1\}$ has a lexicographically lower conjugate vector. We define the weight of $Q$, denoted $w(Q)$, as the sum of the last element of the conjugate vector of $Q$, and the number of occurrences of the string “10” in the conjugate vector of $Q$. Note that in view of the above observation, the color of $w(Q)$ integers in the set \{a, a + d, a + d + 1, \ldots, a + (k-1)d, \ldots, a + (k-1)(d+1)\} can be inferred to be blue.

We now derive an upper bound on $g(2, N, k, 1)$. There are $N^2/(k-1)$ choices for the pair $(a, d)$. Of the $2^{k-1}$ possible conjugate vectors for a quasi-progression with first term $a$ and common difference $d$, let $w_\ell$ be the number of conjugate vectors of weight $\ell$. Let 

$$S_\ell = \sum_{\ell=0}^{[\ell/2]} w_\ell 2^{-\ell}$$

denote the weighted sum of all such vectors of length $\ell$. Clearly, $S_\ell = S_{\ell,0} + S_{\ell,1}$, where $S_{\ell,0}$ and $S_{\ell,1}$ denote the weighted sum of conjugate vectors that begin with 0 and 1 respectively, with $S_{1,0} = 1$ and $S_{1,1} = 1/2$. It is easy to see that $A[S_{\ell-1,0} S_{\ell-1,1}]^T = [S_{\ell,0} S_{\ell,1}]^T$ where 

$$A = \begin{bmatrix} 1 & 1 \\ 1/2 & 1 \end{bmatrix}$$

Since $\lambda_{\max}(A) = 1 + \frac{1}{\sqrt{2}}$, we get 

$$g(2, N, k, 1) < \frac{N^2 2^{N-k+1} \left[ \left(1 + \frac{1}{\sqrt{2}}\right)^k + \left(1 - \frac{1}{\sqrt{2}}\right)^k \right]}{2^{(k-1)}}$$

Thus $g(2, N, k, 1) < 2^N$ for $N = \beta_{2,1}^k$, where $\beta_{2,1} = \sqrt{4 - 2\sqrt{2}} = 1.08239\ldots$ is the smallest positive real root of the equation $y^4 - 8y^2 + 8 = 0$. It follows that $Q_1(2, k) > \beta_{2,1}^k$ yielding a tiny improvement over the lower bound in [5].

In general, since there are $r^N$ $r$-colorings of $[1, N]$ and at most $N^2(n+1)^{k-1}$ $k$-term quasi-progressions of diameter $n$, a lower bound of the form $Q_n(r, k) \geq \left(\frac{\sqrt{r/(n+1)}}{k}\right)^k$ follows immediately from the linearity of expectation. However, this bound is only useful when $n \leq r - 2$. Generalizing the approach outlined earlier,
we represent the conjugate vector of $Q$ as an $r$-ary string, and define the weight $w(Q)$ as the sum of the last element of the conjugate vector of $Q$, and the number of occurrences of strings of length two of the form "xy", counted with multiplicity $m(x, y) = \min(x, n - y)$. (Note that $m(x, y)$ denotes the number of conjugate vectors that are lexicographically lower than the given vector and correspond to quasi-progressions that differ from $Q$ in exactly one element.)

As before, let $S_{t,j}$ denote the weighted sum of conjugate vectors of length $t$ beginning with $j$, $0 \leq j \leq n$, with $S_{1,j} = \alpha^j$ for all $j$ where $\alpha = 1 - \frac{1}{r}$. Then $A[S_{0,0} \cdots S_{t,n}]^T = [S_{t+1,0} \cdots S_{t+1,n}]^T$ where

$$A_{r,n} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha & \alpha & \cdots & \alpha & 1 \\ \alpha^2 & \alpha^2 & \cdots & \alpha & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^n & \alpha^{n-1} & \cdots & \alpha & 1 \end{bmatrix}$$

Note that the $(i,j)^{th}$ entry of the matrix $A_{r,n}$ is $\alpha^{m(i-1,n+1-j)} = \alpha^{\min(i-1,n+1-j)}$. Then $Q_n(r,k) > \beta^k$ where $\beta = \beta_{r,n} = \sqrt{r/\lambda_{\max}(A_{r,n})}$. Note that for each $r$, there are only finitely many values for which $\beta_{r,n} > 1$. The first few such values are shown in the following table.

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**References**


