ARITHMETIC OF $3^t$-CORE PARTITION FUNCTIONS

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Abstract
Let $t \geq 1$ be an integer and $a_{3^t}(n)$ be the number of $3^t$-cores of $n$. We prove a class of congruences for $a_{3^t}(n) \mod 3$ by Hecke nilpotence.

1. Introduction

If $t$ is a positive integer, let $a_t(n)$ be the number of $t$-cores of $n$, that is, the number of partitions of $n$ with no hooks of length divisible by $t$. Then, as shown by Garvan, Kim and Stanton [4], the generating function for $a_t(n)$ is given by the following infinite product:

$$
\sum_{n=0}^{\infty} a_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)^t}{(1-q^n)}.
$$

Some congruence properties were established for small $t$. See for example [5, 6, 9].

For $t$ a power of 2, Hirschhorn and Sellers [5] made the following conjecture:

Conjecture 1. If $t$ and $n$ are positive integers with $t \geq 2$, then for $k = 0, 2$,

$$
a_{2^t}\left(3^{2^t-1}(24n + 8k + 7) - \frac{4^t-1}{3}\right) \equiv 0 \pmod{2}.
$$


Recently, motivated by the work of Nicolas and Serre [11], Bellaïche and Khare [1] studied the structure of the Hecke algebras of modular forms modulo $p$ for all primes $p$, extending the results of Nicolas and Serre for $p = 2$. In particular, in the appendix of [1], Bellaïche and Khare explicitly determined the upper bound of the degree of nilpotence of Hecke algebras modulo 3. It is obvious that we can use their results to study $3^t$-core partition functions.
Theorem 1. Let \( l \) be an integer such that \( l \geq \frac{4^r-1}{3} \). Then for any distinct primes \( \ell_1, \ldots, \ell_\ell \) which are congruent to \( 2(\text{mod}\, 3) \), we have

\[
a_{3^t} \left( \ell_1 \cdots \ell_\ell n - \frac{9^r-1}{8} \right) \equiv 0 (\text{mod}\, 3)
\]

for all \( n \) coprime to \( \ell_1 \cdots \ell_\ell \).

The paper is laid out as follows. In Section 2, we recall the results on nilpotence of Hecke algebras for \( p = 3 \). In Section 3, we prove Theorem 1. In Section 4, we give a result on \( a_{3^t}(n) \) modulo powers of 3. Throughout the paper, we put \( a_\ell(\alpha) = 0 \) if \( \alpha \notin \mathbb{N} \).

2. Hecke Nilpotence

The proof of Theorem 1 relies on Hecke nilpotence of modular forms. We recall some facts on modular forms (see [8] for more). For integers \( k > 0 \), we denote by \( \mathcal{S}_k \) the space of cusp forms of weight \( k \) with integer coefficients on \( SL_2(\mathbb{Z}) \). Moreover, let \( \mathcal{S}_k(\text{mod}\, p) \) denote the modular forms in \( \mathcal{S}_k \) with integer coefficients, reduced modulo \( p \), where \( p \) is a prime number. Throughout this paper, \( p = 3 \). As usual, Ramanujan’s \( \Delta \) function is

\[
\Delta(z) = \eta(z)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n \in \mathcal{S}_{12},
\]

where \( q = e^{2\pi iz} \), and \( z \) is on the upper half of the complex plane. Let \( \ell \) be a prime \( \neq p \). If \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathcal{S}_k \), then the action of the Hecke operator \( T_{\ell,k} \) on \( f(z)(\text{mod}\, p) \) is defined by

\[
f(z)|T_{\ell,k} = \sum_{n=1}^{\infty} c(n)q^n,
\]

where

\[
c(n) = \begin{cases} a(\ell n), & \text{if } \ell \nmid n, \\ a(\ell n) + \ell^{k-1}a(n/\ell), & \text{if } \ell|n. \end{cases}
\]

Based on the work Bellaïche and Khare [1], it is known that the action of Hecke algebras on the spaces of modular forms modulo 3 is locally nilpotent. Let

\[
T_{\ell}' = \begin{cases} T_\ell, & \text{if } \ell \equiv 2(\text{mod}\, 3), \\ 1 + T_\ell, & \text{if } \ell \equiv 1(\text{mod}\, 3). \end{cases}
\]

If \( f(z) \in \mathcal{S}_k \), then there is a positive integer \( i \) with the property that

\[
f(z)|T_{\ell_1}'T_{\ell_2}' \cdots T_{\ell_\ell}' = 0(\text{mod}\, 3)
\]
for every collection of primes $\ell_1, \ldots, \ell_i$, where $\ell_j \neq p$ for $j = 1, \ldots, i$. Suppose that $f(z) \not\equiv 0 \pmod{p}$. We say that $f(z)$ has degree of nilpotence $i$ if there exist primes $\ell_1, \ldots, \ell_{i-1}$ such that $\ell_j \neq 3$ for $j = 1, \ldots, i - 1$ for which

$$f(z) | T_{\ell_1} | T_{\ell_2} \cdots | T_{\ell_{i-1}} \not\equiv 0 \pmod{3}$$

and every collection of primes $p_1, \ldots, p_i$, such that $p_j \neq 3$ for $j = 1, \ldots, i$ for which

$$f(z) | T_{p_1} | T_{p_2} \cdots | T_{p_i} \equiv 0 \pmod{3}. \quad (2.2)$$

We denote by $g_k(n)$ the degree of nilpotence of $\Delta^k(z) \pmod{3}$. To obtain congruences for $a_{3^r}(n)$, we need the upper bound for $g_k(3)$.

**Theorem 2 (Bellaïche and Khare [1]).** For any positive integer $k = \sum r_i a_i 3^i$, with $a_1 \in \{0, 1, 2\}, a_r \neq 0$, we have $g_k(3) \leq \sum r_i a_i 2^i$.

**Corollary 1.** If $k = \frac{9^t - 1}{3}$, we have $g_{\frac{9^{t-1}}{3}}(3) \leq \frac{2^t - 1}{3}$.

**Proof.** This is a consequence of Theorem 2 since $k = \frac{9^t - 1}{3} = 1 + 3^2 + \cdots + 3^{2t-2}$. \qed

### 3. Proof of Theorem 1

We now prove Theorem 1.

**Proof of Theorem 1.** By the definition of $a_t(n)$, we have

$$\sum_{n=0}^{\infty} a_{3^n}(n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{3^n})^3}{(1 - q^n)} = \prod_{n=1}^{\infty} (1 - q^n)^{9^n - 1} \pmod{3}.$$  

From the definition of $\Delta(z)$, it follows that:

$$\sum_{n=0}^{\infty} a_{3^n} \left( \frac{n - \frac{9^t - 1}{3}}{3} \right) q^n = \sum_{n=0}^{\infty} a_{3^n}(n) q^{3n + \frac{9^t - 1}{3}} = \Delta \frac{9^t - 1}{3}(z) \pmod{3}. \quad (3.1)$$

Now by (2.1), the definition of $T_\ell^j$ and Corollary 1, the proof of Theorem 1 is immediate. \qed

**Example 1.** If $t = 1$, then $g_1(3) = 1$. Theorem 1 asserts that for primes $\ell \equiv 2 \pmod{3}$, then

$$a_3 \left( \frac{\ell n - 1}{3} \right) \equiv 0 \pmod{3}$$

for all $n$ coprime to $\ell$. In particular, if $\ell \neq 2$, we choose $n = 3 \ell m + 2$, then

$$a_3 \left( \frac{\ell^2 m + \frac{2\ell - 1}{3}}{3} \right) \equiv 0 \pmod{3} \quad (3.2)$$
for all $m \geq 0$. For $\ell = 2$, we can choose $n = 6m + 5$, then
\[
a_3(4m + 3) \equiv 0 \pmod{3}
\] (3.3)
for all $m \geq 0$. In fact, in [7, Cor. 9], we know that $a_3(n)$ are zero in (3.2) and (3.3). So our results are trivial.

**Example 2.** If $t = 2$, then $g_10(3) = 5$. If $\ell_i \equiv 2 \pmod{3}$ for distinct primes $\ell_i$, then we have
\[
a_9\left(\frac{\ell_1\ell_2\ell_3\ell_4\ell_5 n - 10}{3}\right) \equiv 0 \pmod{3}
\]
for all $n$ coprime to $\ell_1\ell_2\ell_3\ell_4\ell_5$. Suppose we choose $\ell_1 = 2, \ell_2 = 5, \ell_3 = 11, \ell_4 = 17, \ell_5 = 23$, then we know
\[
a_9\left(\frac{43010n - 10}{3}\right) \equiv 0 \pmod{3}
\]
for all $n$ such that $(43010, n) = 1$. Actually, since $\ell_4 = 17 \equiv 8 \pmod{9}$, from Theorem 24 and Lemma 35 in [1] we know that $g_9(T_{17}^2) \leq g_9(f) - 2$. So we have
\[
a_9\left(\frac{1870n - 10}{3}\right) \equiv 0 \pmod{3}
\]
for all $n$ coprime to 1870.

**4. Further Remarks**

In [10], Moon and Taguchi proved the following theorem.

**Theorem 3.** Let $k \geq 1$ be a positive integer. Let $\varepsilon : (\mathbb{Z}/3^a \cdot 4\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Then there exist integers $c \geq 0$ and $e \geq 1$, depending on $k$, $a$ and $\varepsilon$ such that for any modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathcal{M}_k(\Gamma_0(3^a \cdot 4), \varepsilon; \mathbb{Z})$, any integer $j \geq 1$, and any $c + ej$ primes $p_1, p_2, \ldots, p_{c+ej}$ coprime to $p_1, p_2, \ldots, p_{c+ej}$, we have
\[
f(z)|T_{p_1}|T_{p_2}| \cdots |T_{p_{c+ej}} \equiv 0 \pmod{3^j}.
\]
Furthermore, if the primes $p_1, p_2, \ldots, p_{c+ej}$ are distinct, then for any positive integer $m$ coprime to $p_1, p_2, \ldots, p_{c+ej}$, we have
\[
a(p_1p_2 \cdots p_{c+ej}m) \equiv 0 \pmod{3^j}.
\]
Since it is easy to see that [12, Theorem 1.64]
\[
\sum_{n=0}^{\infty} a_{3^j}(n)q^{3n+\frac{2^j-1}{3}} = \frac{\eta(3^{j+1}z)^{3^j}}{\eta(3z)}
\]
belongs to $\mathcal{M}_{3^t-1}(\Gamma_0(3^{t+1}), \chi)$ where $\chi(d) = \left(\frac{-1}{d^t} \frac{3^{t+1}-d}{3}ight)$, we have the following theorem.

**Theorem 4.** There exist integers $c \geq 0$ and $e \geq 1$, depending on $t$ such that for any positive integer $j$ and any distinct $c + ej$ primes $p_1, p_2, \ldots, p_{c+ej} \equiv -1(\text{mod } 12)$, we have

$$a_{3^t} \left(\frac{p_1 p_2 \cdots p_{c+ej} n - \frac{3^{t+1}-1}{3}}{3}\right) \equiv 0(\text{mod } 3^t)$$

for any positive integer $n$ coprime to $p_1, p_2, \ldots, p_{c+ej}$.

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**References**


