SYMMETRY IN MAXIMAL \((s - 1, s + 1)\) CORES

Rishi Nath

Department of Mathematics and Computer Science, York College, City University of New York, Jamaica, New York
rnath@york.cuny.edu

Received: 11/2/14, Revised: 9/17/15, Accepted: 3/5/16, Published: 3/11/16

Abstract
Let \(s\) be even and greater than 2. We explain a “curious symmetry” for maximal \((s - 1, s + 1)\)-core partitions first observed by T. Amdeberhan and E. Leven. Specifically, using the \(s\)-abacus, we show such partitions have empty \(s\)-core and that their \(s\)-quotient is comprised of \(2\)-cores. These conditions impose strong conditions on the partition structure, and imply both the Amdeberhan-Leven result and additional symmetry. We conclude by finding the most general family of partitions that exhibit these symmetries, and obtain some new results on maximal \((s - 1, s, s + 1)\)-core partitions.

1. Introduction
The study of simultaneous core partitions, which began fifteen years ago, has seen recent interest due mainly to a conjecture of Armstrong on the average size of an \((s, t)\)-core when \(\gcd(s, t) = 1\). R. Stanley and F. Zanello [19] verified the Armstrong conjecture when \(t = s + 1\); they employ a certain partially ordered set \(P_{s,s+1}\) associated to the set of simultaneous \((s, s+1)\)-cores which (in this case) exhibits well-understood symmetry. However this poset approach appears difficult to generalize, and P. Johnson [14] recently settled the general case of the Armstrong conjecture using methods from Erhart theory. Amdeberhan and Leven [5] deviate slightly to examine \((s - 1, s + 1)\)-cores in the case where \(s\) is even and greater than 2. They do not prove the Armstrong conjecture in this case; they do however explore a “curious symmetry” for the poset \(P_{s-1,s+1}\). Our Theorem 6 states their result.

Hidden by the Amdeberhan-Leven proof (which involves integral and fractional parts of a real number) is a connection with the \(s\)-core and \(s\)-quotient structure viewed on an \(s\)-abacus. From this vantage point, the Amdeberhan-Leven theorem is a consequence of symmetry in each runner of the \(s\)-abaci of the maximal \((s - 1, s + 1)\)-core; it also reveals additional symmetry in each row not picked up
Figure 1: The 8-abacus of \( \kappa_{7,9} \)

by the Amdeberhan-Leven formulation. The \( s \)-abaci of maximal \((s - 1, s + 1)\)-cores also provide a convenient link to the study of maximal \((s - 1, s, s + 1)\)-cores, objects of recent interest.

We introduce some notation to state the main theorem. Given a partition \( \lambda \), let \( \lambda^0 \) be its \( s \)-core and \((\lambda(0), \lambda(1), \ldots, \lambda(s-1))\) be its \( s \)-quotient. Let \( \kappa_{s \pm 1} \) be the unique maximal simultaneous \((s - 1, s + 1)\)-core partition and \( \tau_\ell = (\ell, \ell - 1, \ell - 2, \ldots, 1) \) be the \( \ell \)-th 2-core partition.

**Theorem 1.** Let \( s = 2k > 2 \). Then \( \kappa_{s \pm 1} \) has the following \( s \)-core and \( s \)-quotient structure:

1. \((\kappa_{s \pm 1})^0 = \emptyset\).
2. \( \kappa_{s \pm 1}(i) = \kappa_{s \pm 1}(s - 1 - i) = \tau_{k - 1 - i} \) where \( 0 < i < k - 1 \).

**Example 2.** The 8-abacus of \( \kappa_{7,9} \) and the associated 8-quotient are shown in Figure 1 and Figure 2 respectively. Note: the 8-quotient consists of a sequence of 2-core partitions encoded in the runners of the 8-abacus.

The basic definitions are covered in Section 2. In Section 3.1 we describe the \( s \)-abacus of \( \kappa_{s \pm 1} \), which we use to prove Theorem 1. We provide an alternate proof of the Amdeberhan-Leven result in Section 3.2. In Section 4.1 we demonstrate an additional symmetry in the rows of the \( s \)-abacus of \( \kappa_{s - 1, s + 1} \) and describe the most general family of partitions which satisfy the symmetries exhibited by \( \kappa_{s \pm 1} \) in Section 4.2. In Section 5.1 we offer a characterization of the \( s \)-abacus of a maximal \((s - 1, s, s + 1)\)-core and examine the relationship between maximal \((s - 1, s + 1)\) and \((s - 1, s, s + 1)\)-cores, when \( s \) is even and greater than 2. We conclude with some questions in Section 5.2.
2. Preliminaries

2.1. Basic Definitions

Let \( \mathbb{N} = \{0, 1, \ldots \} \) and \( n \in \mathbb{N} \). A partition \( \lambda \) of \( n \) is defined as a finite, non-increasing sequence of positive integers \( (\lambda_1, \lambda_2, \ldots, \lambda_k) \) that sums to \( n \). Each \( \lambda_\alpha \) is known as a component of \( \lambda \). Then \( \sum_\alpha \lambda_\alpha = n \), and \( \lambda \) is said to have size \( n \), denoted \( |\lambda| = n \).

The Young diagram \( [\lambda] \) is a graphic representation of \( \lambda \) in which rows of boxes corresponding to the integer values in the partition sequence are left-aligned. Then \( \lambda^* \) is the conjugate partition of \( \lambda \) obtained by exchanging rows and columns of the Young diagram of \( \lambda \); \( \lambda \) is self-conjugate if \( \lambda = \lambda^* \). Using matrix notation, a hook \( h_{\iota\gamma} \) of \( [\lambda] \) with corner \( (\iota, \gamma) \) is the set of boxes to the right of \( (\iota, \gamma) \) in the same row, below \( (\iota, \gamma) \) in the same column, and \( (\iota, \gamma) \) itself. Given \( h_{\iota\gamma} \), its length \( |h_{\iota\gamma}| \) is the number of boxes in the hook. The set \( \{h_{11}\} \) are the first-column hooks of \( \lambda \).

One can remove a hook \( h \) of \( \lambda \) by deleting boxes in \( [\lambda] \) which comprise \( h \) and migrating any remaining detached boxes up and to the left. In this way a new partition \( \lambda' \) of size \( n - |h_{\iota\gamma}| \) is obtained. An \( s \)-hook is a hook of length \( s \). An \( s \)-core partition \( \lambda \) is one in which no hook of length \( s \) appears in the Young diagram.

Example 3. Let \( \lambda = (4, 3, 2) \). Then the Young diagram of \( \lambda \) is shown in Figure 3. Note that \( h_{21} \) is of length 4.

2.2. Simultaneous \((s, t)\)-core Partitions

Let \( s \) and \( t \) be positive integers. A simultaneous \((s, t)\)-core partition is one in which no hook of length \( s \) or \( t \) appears. In 1999, J. Anderson [6] proved when \( \gcd(s, t) = 1 \), there are exactly \( \frac{(s^2-1)(t^2-1)}{24} \) simultaneous \((s, t)\)-cores.

Subsequent work by B. Kane [1], J. Olsson and D. Stanton [18], and J. Vandehey [21] confirmed the existence of a unique maximal \((s, t)\)-core of size \( \frac{(s^2-1)(t^2-1)}{24} \), denoted by \( \kappa_{s,t} \), which contains all others. Here maximal is meant in terms of size; containment is being able to fit the Young diagram of one partition inside another.

Theorem 4. (Olsson-Stanton, Theorem 4.1, [18]) Let \( \gcd(s, t) = 1 \). There is a unique maximal simultaneous \((s, t)\)-core \( \kappa_{s,t} \) of size \( \frac{(s^2-1)(t^2-1)}{24} \). In particular, \( \kappa_{s,t} \) is self-conjugate.
Theorem 5. (Vandehey, [21]) Let $\gcd(s, t) = 1$. Then $\kappa_{s,t}$ contains all other $(s, t)$-cores.

We note that A. Tripathi [20] and M. Fayers [9] obtained some of the above results using different methods.

A paper of D. Armstrong, C. Hanusa and B. Jones [7] includes a conjecture (the Armstrong conjecture) that the average size of an $(s, t)$-core is $\frac{(s+t+1)(s-1)(t-1)}{24}$. Stanley and Zanello [19] resolved this conjecture for the case $t = s + 1$ by employing a bijection between lower ideals in the poset $P_{s,s+1}$ and simultaneous $(s, s+1)$-cores. We outline this bijection for general $s$ and $t$. Let $P_{s,t}$ be the partially ordered set whose elements are all positive integers not contained in the numerical semigroup generated by $s, t$. The partial order requires $z_1 \in P_{s,t}$ to cover $z_2 \in P_{s,t}$ if $z_1 - z_2$ is either $s$ or $t$. Under this map, a lower ideal $I$ of $P_{s,t}$ corresponds to an $(s, t)$-core partition whose first-column hook lengths are exactly the values in $I$. Then $P_{s,t}$ itself corresponds to $\kappa_{s,t}$.

The Armstrong conjecture was verified for self-conjugate partitions by W. Chen, H. Huang, and L. Wang [8] and for $(s, ms + 1)$-cores by A. Aggarwal [2] before a full proof was given by P. Johnson [14] using Erhart theory. Since then, V. Wang [22], using an approach of M. Fayers [10], has found a proof of the Armstrong conjecture that avoids Erhart theory; Wang also settles a generalization of the Armstrong conjecture due to M. Fayers [11].

Simultaneous core partitions have also generated interest outside of the Armstrong conjecture. For example, Aggarwal has also proved a partial converse to a theorem of Vandehey on the containment of simultaneous $(r, s, t)$-cores [3] for distinct $r, s, t$. In another direction, Amdeberhan [4] proposed several conjectures on maximal $(s - 1, s + 1)$-cores; these have been proved, first by J. Yang, M. Zhong and R. Zhou [24] and later by H. Xiong [23]. We discuss these developments in Section 5.
2.3. A “Curious Symmetry”

For $s$ even, Amdeberhan and Leven examine $P_{s-1,s+1}$ via a rectangle $R$ with $s - 2$ rows and $s$ columns, constructed as follows: the bottom-left corner is labelled by 1, the numbers increase from left-to-right and bottom-to-top, and the largest position, in the upper-right corner, is labeled by $(s - 2)(s)$. If $x \in P_{s-1,s+1}$ then $x$ is entered into this rectangle; otherwise, the position is left blank. Positions are labeled by pairs $(i,j)$, where $i$ enumerates columns from left-to-right ($1 \leq i \leq s$), and $j$ rows from bottom-to-top ($1 \leq j \leq s - 2$). They then prove the following result, which they call a “curious symmetry.”

**Theorem 6.** (Amdeberhan-Leven, Theorem 2.2, [5]) For $s \geq 4$ and even, the $(i,j)$ entry of $R$ is an element of $P_{s-1,s-1}$ if and only if $(i,s - 2 - j)$ is not. Equivalently, for $1 \leq i \leq s$ and $1 \leq j \leq s - 2$, $i + s(j - 1) \in P_{s-1,s+1}$ if and only if $i + s(2 - j) \notin P_{s-1,s+1}$.

There is a precedent for the case Amdeberhan-Leven consider. For $s = 2k > 1$, the maximal simultaneous $(s - 1,s + 1)$-core is self-conjugate, by Theorem 4. In [12] C. Hanusa and the author showed that it is more natural to think about self-conjugate $(s - 1,s + 1)$-core partitions than self-conjugate $(s,s + 1)$-cores (the latter of which are better behaved in the non-self-conjugate case).

We now review the $s$-abacus, $s$-core, and $s$-quotient constructions.

2.4. Bead-sets

The first column hook lengths uniquely determine a partition $\lambda$. We can generalize the set of first column hooks using the notation of a bead set $X$ corresponding to $\lambda$, where $X = \{0, \cdots, k - 1, |h_{11}| + k, |h_{21}| + k, |h_{31}| + k, \cdots \}$ for some non-negative integer $k$. It can also be seen as a finite set of non-negative integers, represented by beads at integral points of the $x$-axis, i.e. a bead at position $x$ for each $x$ in $X$ and spacers at positions not in $X$. Then $|X|$ is the number of
beads that occur after the zero position, wherever that may fall. We say \( X = \{ 0, \ldots, k - 1, |h_{11}| + k, |h_{21}| + k, |h_{31}| + k, \ldots \} \) is normalized with respect to \( s \) if \( k \) is the minimal integer such that \( |X| \equiv 0 \pmod{s} \). The minimal bead-set \( X \) of \( \lambda \) is one where \( 0 \) labels the first spacer, and is equal to the set of first column hook lengths.

**Example 7.** Suppose \( \lambda = (4, 3, 2) \). Then \( \{h_{11}\} = \{2, 4, 6\} \) is the set of first column hook lengths, and a minimal bead set. Then \( X' = \{0, 2+1, 4+1, 6+1\} = \{0, 3, 5, 7\} \) and \( X'' = \{0, 1, 2, 3, 2 + 4, 4 + 4, 6 + 4\} = \{0, 1, 2, 3, 6, 8, 10\} \) are two bead sets that also correspond to \( \lambda \). \( X' \) and \( X'' \) are normalized with respect to 4 and 7 respectively, since \( X' \equiv 0 \pmod{4} \) and \( X'' \equiv 0 \pmod{7} \).

### 2.5. 2-cores and Staircase Partitions

The results in this section are stated without proof; for more details see Section 2 in [17]. The set of hooks \( \{h_{11}\} \) of \( \lambda \) correspond bijectively to pairs \((x, y)\) where \( x \in X, y \notin X \) and \( x > y \); that is, a bead in a bead-set \( X \) of \( \lambda \) and a spacer to the left of it. Hooks of length \( s \) are those such that \( x - y = s \).

Bead \( x \) in the minimal bead-set \( X \) are in bijection with components of \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \). The following result, which appears as in Lemma 2.4 in [17], allows us to recover the size of the component from its corresponding bead.

**Lemma 1.** Let \( X \) be a bead set of a partition \( \lambda \). The size of the component \( \lambda_\alpha \) of \( \lambda \) corresponding to the bead \( x' \in X \) is the number of spacers to the left of the bead, that is, \( \lambda_\alpha = |y \notin X : y < x'| \).

Let \( \tau_\ell = (\ell, \ell - 1, \ell - 2, \cdots, 1) \) be the \( \ell \)th staircase partition. Then \( |\tau_\ell| = t_\ell \) where \( t_\ell = \binom{\ell + 1}{2} \) (the \( \ell \)th triangular number). The following two lemmas are well-known.

**Lemma 2.** The 2-core partitions are exactly the staircase partitions.
Lemma 3. The minimal bead set $X$ for the 2-core $\tau_\ell$ is $\{1, 3, 5, \ldots, 2\ell - 3, 2\ell - 1\}$. In other words, the 2-core partitions are a sequence of alternating spacers-and-beads of length $2\ell - 1$.

2.6. The $s$-abacus

Given a fixed integer $s$, we can arrange the nonnegative integers in an array of columns and consider the columns as runners:

\[
\begin{array}{cccc}
ms & ms + 1 & (m + 1)s - 1 & ms - 1 \\
(m - 1)s & \vdots & \\
s & s + 1 & 2s - 1 & \\
0 & 1 & \cdots & s - 1
\end{array}
\]

The column containing $i$ for $0 \leq i \leq s - 1$ will be called runner $i$. The positions $0, 1, 2, \cdots$ on the $i$th runner corresponding to $i, i + s, i + 2s, \cdots$ will be called row positions on runner $i$. Consider a bead set $X$. Placing a bead at position $x$ for each $x \in X$ gives the $s$-abacus diagram of $X$. Positions not occupied by beads are spacers. A normalized abacus will be one whose bead set $X$ is normalized, and the minimal abacus one in which $X$ is minimal (or, the first spacer labels the zero position). Note that a bead $x$ in runner $i$ with a spacer $y$ one row below, but also in runner $i$, corresponds to an $s$-hook of $\lambda$. The following is immediate.

Lemma 4. An $s$-abacus in which no spacer appears directly below a bead on the same runner corresponds to an $s$-core partition.

2.7. The $s$-core and $s$-quotient

By removing a sequence of $s$-hooks from $\lambda$ until no $s$-hooks remain, one obtains its $s$-core $\lambda^0$. The $s$-abacus of $\lambda^0$ can be found from the $s$-abacus of $\lambda$ by pushing beads in each runner down as low as they can go (see Theorem 2.7.16, [13], with changed orientation). Hence $\lambda^0$ is unique since it is independent of the way the $s$-hooks are removed. For $0 \leq i \leq s - 1$ let $X_i = \{j : i + js \in X\}$ and let $\lambda_{(i)}$ be the partition represented by the bead-set $X_i$. The $s$-quotient of $\lambda$ is the sequence $(\lambda_{(0)}, \cdots, \lambda_{(s-1)})$ obtained from $X$. The next lemma is Proposition 3.5 in [17].

Lemma 5. Let $\lambda$ be a partition with $s$-core $\lambda^0$ and $s$-quotient $(\lambda_{(i)}), 0 \leq i \leq s - 1$. Then

1. Every 1-hook in $\lambda_{(i)}$ corresponds to an $s$-hook in $\lambda$ for $0 \leq i \leq s - 1$.

2. $n = |\lambda^0| + s \cdot \sum_{i} |\lambda_{(i)}|$. 


We note that $X_i$ could consist of an interval $[0, m]$ and thus $\lambda_{(i)}$ would be empty (as is the case with $\lambda_{(3)}$ and $\lambda_{(4)}$ in Example 1.2). Lemma 5 implies that there exists a bijection between a partition $\lambda$ and its $s$-core and $s$-quotient, such that each node in some $\lambda_{(i)}$ corresponds to an $s$-hook in $\lambda$. The situation is strengthened when $\lambda$ is self-conjugate.

**Lemma 6.** Suppose $|X| = 0 \pmod{s}$. Let $\lambda^*$ be the conjugate of $\lambda$, $(\lambda^*)^0$ its $s$-core and let $(\lambda^*_{(i)})$ be the $s$-quotient of $\lambda^*$, $0 \leq i \leq s - 1$. Then

1. $(\lambda^*)^0 = (\lambda^0)^*$
2. $(\lambda_{(i)})^* = \lambda_{(s-1-i)}^*$

In particular, $\lambda = \lambda^*$ if and only if $\lambda^0 = (\lambda^0)^*$ and $(\lambda_{(i)})^* = (\lambda^*)_{(i)}$.

### 2.8. The Axis $\theta(X)$ of a Bead Set of $\lambda$

The following results and their proofs can be found in Section 4, [15].

**Proposition 1.** Suppose $\lambda$ is a partition of $n$ and let $X$ be a bead-set for $\lambda$. Then there exists a half-integer $\theta(X)$ (that is, an element of $\mathbb{Z} + \frac{1}{2}$) such that the number of beads to the right of $\theta(X)$ equals the number of spaces to its left. Conversely, given a bead-spacer sequence and a half-integer such that the number of beads to the right equals the number of spaces to the left, one can recover a partition $\lambda$.

Although the number of beads to the right of $\theta(X)$ and the number of spacers of the left of $\theta(X)$ remain unchanged on any bead-spacer sequence associated to $\lambda$, the half-integer value assigned to $\theta(X)$ will depend on $X$.

**Example 8.** Consider $\lambda = (4, 3, 2)$. Then $\theta(X) = 2.5$ and $\theta(X') = 3.5$ for $X = \{2, 4, 6\}$ and $X' = \{0, 3, 5, 7\}$ respectively. See Figure 6 and Figure 7.

We call $\theta(X)$ the axis of a bead set $X$ of $\lambda$. Each $X_i$ has an axis $\theta(X_i)$; when $\lambda^0 = \emptyset$, the value does not change as $i$ runs from $0$ to $s - 1$.

**Lemma 7.** Suppose $X$ is normalized with respect to $s$ and $|X| = ms$. Then the following are equivalent:

1. $\lambda^0 = \emptyset$
2. each runner has exactly $m$ beads
3. $\theta(X_i) = m - \frac{1}{2}$ for all $0 \leq i \leq s - 1$.

**Example 9.** The maximal $(5, 7)$-core $\kappa_{5,7}$ has empty 8-core. In the normalized (minimal) 8-abacus in Figure 1, $|X| = 3 \cdot 8$, whereas each runner has 3 beads, and $X_i$ has axis $\theta(X_i) = \frac{5}{2}$ for $0 \leq i \leq 7$. 
If $\lambda$ is self-conjugate we say $X$ has an axis of symmetry.

**Corollary 1.** Let $X$ be a bead-set for $\lambda$. Then $\lambda$ is a self-conjugate partition if and only if there exists a half-integer $\theta(X)$ such that beads and spacers in $X$ to the right of $\theta(X)$ are reflected respectively to spacers and beads in $X$ to its left.

**Example 10.** The maximal $(3,5)$-core $\kappa_{3,5}$ is self-conjugate, with minimal bead set $X = \{1, 2, 4, 7\}$, and $\theta(X) = 3.5$. Then beads and spacers to the right of 3.5 are reflected to spacers and beads to the left of it. See Figure 8.

Note that when a bead set is not minimal, the sequence of beads in positions $[0, 1, \cdots, k]$ will be reflected onto spacers in positions greater than the last bead.

### 3. The Structure of $\kappa_{s\pm 1}$

#### 3.1. The $s$-abacus $\alpha(s)$

To recover the Amdeberhan-Leven result we first construct the $s$-abacus of $\kappa_{s\pm 1}$.

**Definition 11.** Let $s = 2k > 2$, and consider $s$ runners, indexed from left-to-right by $0 \leq i \leq s - 1$ and $s - 2$ rows, indexed from bottom-to-top by $0 \leq j \leq s - 3$. We construct the $s$-abacus $\alpha(s)$ as follows: for each $i \in [0, k - 2]$, runners $i$ and $s - i - 1$ are composed of beads in the first $i$ rows; spacers-and-beads alternate in rows $j > i$ until the total number of beads in each runner is $k - 1$. Spacers fill the remainder of the rows.

**Example 12.** Consider the 8-abacus $\alpha(8)$. It has three beads in each runner. Runners 3 and 4 consist of three beads below three spacers; runners 2 and 5 have two beads followed by a spacer-and-bead, then two spacers; runners 1 and 6 have one bead followed by spacer-bead-spacer-bead-spacer; and runners 0 and 7 have an alternating sequence of spacers-and-beads. [See Figure 1.]

**Lemma 8.** The $s$-abacus $\alpha(s)$ is normalized with respect to $s$.

**Proof.** The total number of beads in $\alpha(s)$ is $2k(k - 1) = s(s - 2)$. □

**Lemma 9.** The following holds for the $s$-abacus $\alpha(s)$
1. There is a bead in row \( j \) of runner 0 if and only if there is a bead in row \( j - 1 \) of runner 1. 

2. There is a bead in row \( j \) of runner \( 2k - 1 \) if and only if there is a bead in row \( j - 1 \) of runner \( 2k - 2 \). 

3. There is a spacer in row \( j \) of runner 0 if and only if there is a spacer in row \( j + 1 \) of runner 1. 

4. There is a spacer in row \( j \) of runner \( 2k - 1 \) if and only if there is a spacer in row \( j + 1 \) of runner \( 2k - 2 \). 

Proof. By Definition 11, runner \( i = 0 \) begins in row \( j = 0 \) with a spacer, and continues upwards with alternating beads-and-spacers. Runner \( i = 1 \) begins with a bead in row 1, and continues upwards, alternating spacers-and-beads. Since both columns have \( 2k - 2 \) rows, (1) and (3) follow. For (2) and (4), a similar argument holds. \( \square \)

Lemma 10. The \((s + 2)\)-abacus \( \alpha(s + 2) \) can be obtained from the \( s \)-abacus \( \alpha(s) \) using the following procedure: 

1. Append a new row of \( 2k \) beads below \( \alpha(s) \). 

2. Append a new row of \( 2k \) spacers above \( \alpha(s) \). 

3. Append a new runner of length \( 2k - 2 \) consisting of alternating beads-and-spacers to the left, and an identical column to the right, of \( \alpha(s) \). [Both of these columns start with a bead in the bottom row.] 

4. Append a single spacer to the bottom, and a single bead at the top of, both new runners in step (3). [The total number of beads in all runners, both the two new runners, as well as the \( s = 2k \) previous runners, will now be \( k \).] 

5. Renumber the runners with \( i' \) so \( 0 \leq i' \leq 2k + 1 \) and the rows with \( j' \) so that \( 0 \leq j' \leq 2k - 1 \). Renumber the abacus positions, with 0 in the bottom left-most corner, increasing from left-to-right and bottom-to-top, with final position \((2k + 1)(2k - 1)\) in the upper-right-hand corner. 

Proof. It is enough to see that the result of these five steps satisfies Definition 11 for \( \alpha(s + 2) \). \( \square \)

Example 13. To see how Lemma 10 is used to obtain \( \alpha(10) \) from \( \alpha(8) \), see Appendix A, Figure 12 and Figure 13.
Recall $\lambda^0$ and $(\lambda(i))$ for $0 \leq i \leq s - 1$ are the $s$-core and $s$-quotient of $\lambda$ respectively, and that $\tau_{\ell}$ is the $\ell$th 2-core partition. For the following two lemmas we abuse notation and let $\alpha(s)$ refer to both the $s$-abacus and its corresponding partition.

Lemma 11. Suppose $s = 2k > 2$. Then

1. $\alpha(s)^0 = \emptyset$
2. $\alpha(s)(i) = \alpha(s)(s-i-1) = \tau_{k-i+1}$.

Proof. We prove each condition separately.

1. Since each runner $\alpha(s)_i$ has $k - 1$ beads and $k - 1$ spacers, the removal of all $s$-hooks throughout all the runs will result in an $s$-abacus with each runners having $k - 1$ beads beneath $k - 1$ spacers. This arrangement corresponds to the empty partition.

2. We use induction on $k$. For $k = 2$ it is true. Assume it is for $k$. We obtain the $\alpha(s+2)$ from $\alpha(s)$ by Lemma 11. By construction, for $1 \leq i' \leq 2k$ we have $|\alpha(s)(i'-1)| = |\alpha(s+2)(i')|$: hence, by the inductive hypothesis and since $i + 1 = i'$, $|\alpha(s+2)(i')| = \tau_{(k+1)-i'-1}$. It only remains to check $i' = 0$ and $2k+1$. The proof is finished using Lemma 3, and (3) and (4) of Lemma 10.

Example 14. $\alpha(8)$ has 8-quotient $(\lambda(0), \cdots, \lambda(s-1)) = ((3, 2, 1), (2, 1), (1), \emptyset, (1), (2, 1), (3, 2, 1))$.

[See Appendix A, Figure 12 and Appendix B, Figure 16.]

Recall that when $\tau_{\ell}$ is the $\ell$th 2-core partition, we let $t_{\ell} = |\tau_{\ell}|$.

Lemma 12. Let $s = 2k > 2$. Then $\alpha(s)$ is the minimal $s$-abacus for $\kappa_{s-1, s+1}$.

Proof. By construction, $\alpha(s)$ is minimal, since zero labels the first spacer. We must show:

1. $|\alpha(s)| = \frac{(2k-1)^2-1)((2k+1)^2-1)}{24}$
2. $\alpha(s)$ contains no $(2k-1)$-hooks or $(2k+1)$-hooks.

Then by the uniqueness implied by Theorem 4, $\alpha(s) = \kappa_{s-1, s+1}$. We use the structure of $\alpha(s)$ and induction on $k$.

By Lemma 5, $|\lambda| = |A^0| + s \cdot \sum |\lambda(i)|$. Since $\alpha(s)^0 = \emptyset$, to prove (1), it is enough to calculate $2k \cdot \sum_i |\alpha(s)(i)|$, which equals $2k \cdot 2 \sum_{i=1}^{k-1} t_i = (4k)^2 \frac{(k-1)(k)(k+1)}{6}$. In
particular $4k^4(\frac{k^2-1}{6}) = \frac{4k^4-16k^2}{24} = \frac{(4k^2-4k)(4k^2+4k)}{24}$. Finally, after completing-the-square, one obtains

$$\frac{(2k-1)^2 - 1)((2k+1)^2 - 1)}{24}.$$ 

To prove (2), we use induction on $k > 2$. For the basic case, $s=4$, it holds: $\alpha(4)$ has no 3-hooks or 5-hooks. [See Appendix A, Figure 10.] By the inductive hypothesis we know the $2k$-abacus of $\kappa_{2k\pm 1}$ contains no $(2k-1)$-hooks or $(2k+1)$-hooks. More specifically, no bead in $\alpha(s)$ has a space either $2k+1$ or $2k-1$ positions below it. Apply Lemma 10 to obtain $\alpha(s+2)$; this adds two additional positions between the beads and spacers arising from $\alpha(s)$. Hence there are no $(2k+1)$-hooks or $(2k+3)$-hooks arising from bead-spacer pairs $(x, y)$ where both $x$ and $y$ are in runners $1 < i' < 2k-2$. It remains to examine the beads and spacers introduced by runners $i' = 0, 2k+1$.

If a bead in row $j'$ of runner $i' = 0$ were to add a new $(2k+3)$-hook, a spacer would have to appear in row $j' - 2$ of the runner $i' = 2k+1$. By construction, such positions are occupied by beads, since runners 0 and $2k+1$ are identical. If a bead in row $j'$ of $i' = 0$ were to add a new $(2k+1)$-hook, a spacer would have to appear in row $j' - 1$ of runner $i' = 1$; by the Lemma 9(1), this position is always occupied by a bead.

If a bead in row $j'$ on runner $i' = 2k+1$ were to add a new $(2k+3)$-hook, a spacer would appear in row $j' - 1$ of runner $i' = 2k$; by Lemma 9(2) this position is always occupied by a bead. If a bead in row $j'$ of runner $i' = 2k+1$ were to add a new $(2k+1)$-hook, a spacer would have to appear in the same row in the runner $i' = 0$. By construction, the two runners are identical, so a bead in one implies a bead in the other.

If a spacer in row $j'$ of runner $i' = 0$ were to add a new $(2k+3)$-hook, a bead would have to appear in row $j'+1$ of runner $i' = 1$; by Lemma 9(3), this position is always occupied by a spacer. If a spacer in row $j'$ of $i' = 0$ were to add $(2k+1)$-hook, a bead would have to appear in the same row of runner $i' = 2k+1$. By construction, the two runners are identical, so a spacer in one implies a spacer in the other.

If a spacer in row $j'$ of runner $i' = 2k+1$ were to add a new $(2k+3)$-hook, a bead would have to appear in row $j'+2$ in runner $i' = 0$; by construction, since both runners are identical alternating sequences of spacer-and-beads, such positions are occupied by spacers. If a spacer in row $j'$ of runner $i' = 2k+1$ were to add a new $(2k+1)$-hook, a bead would have to appear in row $j'+1$ of runner $i' = 2k$; by Lemma 9(4) this position is occupied by a spacer.
3.2. An Alternative Proof of Amdeberhan-Leven

Using the results of the previous section, and a few lemmas, we can provide an alternative proof to Theorem 6. We begin with a classical result of Sylvester.

**Lemma 13.** The largest integer in $P_{s,t}$ is $st - s - t$.

Let $R$ be the rectangle described in Section 2.3.

**Corollary 2.** $R$ does not contain 0 or $s^2 - 2s$.

**Proof.** $R$ does not contain 0 by construction. By Lemma 13, $s^2 - 2s - 1$ is the largest integer in $P_{s-1,s+1}$, hence also in $R$. □

**Definition 15.** We say $(i, j) \in \alpha(s)$ if $i + js \in P_{s-1,s+1}$ where $0 \leq i \leq s - 1$ and $0 \leq j \leq s - 3$.

**Lemma 14.** Let $0 \leq i \leq s - 1$ and $0 \leq j \leq s - 3$. Then $(i, j) \in \alpha(s)$ if and only if $i + js \in R$.

**Proof.** By Lemma 12 $\alpha(s)$ is the minimal $s$-abacus for $\kappa_{s \pm 1}$. By the discussion in Section 2.7, it contains exactly the same values as $P_{s \pm 1}$. Since $R$ has the same values as $P_{s \pm 1}$ by construction, we are done. □

**Proof of Theorem 6.** By Lemma 14 the contents of $R$ and $\alpha(s)$ are identical; in particular by Corollary 2 we do not lose anything by inserting 0 and removing $s^2 - 2s$ from the diagram. This has the effect of shifting the rightmost column of $R$ to the first column of $\alpha(s)$ and up one row. Hence it is enough to prove the following condition: $(i, j) \in \alpha(s)$ if and only if $(i, s - 3 - j) \notin \alpha(s)$ for $0 \leq i \leq s - 1$ and $0 \leq j \leq s - 3$.

By induction on $k$. For $k = 2$ it is clear. Suppose $(i, j) \in \alpha(s)$ if and only if $(i, s - 3 - j) \notin \alpha(s)$ holds for $s = 2k$. Consider now $s = 2(k + 1)$. The inductive hypothesis and Lemma 10 imply that $(i', j') \in \alpha(s)$ if and only if $(i', s - 1 - j') \notin \alpha(s)$ for $1 \leq i' \leq s$ and $1 \leq j' \leq s - 2$. It remains to show the property holds for $(i, j)$ when $j' = 0$ or $s - 1$ and when $i = 0$ or $s + 1$. However by construction when $j' = 0$ and $1 \leq i' \leq s$, $(i', 0) \in \alpha(s)$ and $(i', s - 1) \notin \alpha(s)$. When $i = 0$ or $s + 1$, the runner consists of alternating sequence of spacers-and-beads, hence the property holds. □

4. Generalizations

4.1. Additional Symmetry for Maximal $(s - 1, s + 1)$-cores

Using Theorem 1 we can strengthen the Amdeberhan-Leven result to include additional symmetry.
Theorem 16. Let \( s = 2k > 2 \) and let \( \alpha(s) \) be the \( s \)-abacus of \( \kappa_{s \pm 1} \). Let \( 0 \leq i \leq s - 1 \) and \( 0 \leq j \leq s - 3 \). Then the following are equivalent:

1. \( (i, j) \in \alpha(s) \)
2. \( (i, s - 3 - j) \notin \alpha(s) \)
3. \( (s - 1 - i, j) \in \alpha(s) \).

Proof. By Theorem 6 it is sufficient to prove \( 1 \iff 3 \). This follows from Lemmas 9, 11, and 12, and an induction argument similar to the proof Theorem 6.

4.2. \((UD, -)\) and \((RL, +)\) Symmetry

The symmetries exhibited by the \( s \)-abacus of \( \kappa_{s \pm 1} \) can be formalized and generalized to a larger family of partitions. For the remainder of this section we assume that the bead-set \( X \) of \( \lambda \) is normalized with respect to \( s \). Let \( 0 \leq i \leq s - 1 \). Suppose that the \( s \)-abacus of \( \lambda \) has maximum value \( i + (q - 1)s \). In particular, the normalized \( s \)-abacus of \( \lambda \) has \( s \) columns and \( q \) rows, indexed by pairs \( (i, j) \) where \( 0 \leq j \leq q - 1 \).

Definition 17. We say the \( s \)-abacus of \( \lambda \) exhibits \((UD, -)\) symmetry if there is a bead in the \((i, j)\) position if and only if there is a spacer in the \((i, q - 1 - j)\) position. \([UD\) here refers to up-down.\]

Lemma 15. The \( s \)-abacus of \( \lambda \) exhibits \((UD, -)\) symmetry if and only if \( q \) is even, \( \lambda(i) = \lambda^*_i \) for all \( 0 \leq i \leq s - 1 \), and \( \lambda^0 = \emptyset \).

Proof. Suppose the \( s \)-abacus \( \pi \) of \( \lambda \) exhibits \((UD, -)\) symmetry. Then if \( (i, j) \in \pi \) if and only if \( (i, q - 1 - j) \notin \pi \). This is equivalent to each runner \( i \) having axis \( \theta(X_i) = \frac{q - 1}{2} \) such that beads and spacers less than \( \theta(X_i) \) are reflected across to spacers and beads. Hence \( q \) must be even, so beads and spacers can be paired. By Corollary 1, this also implies that \( \lambda(i) = \lambda^*_i \) for each \( 0 \leq i \leq s - 1 \). Finally, by Lemma 7, \( \lambda^0 = \emptyset \). The proof in the other direction is clear. \( \square \)
Definition 18. We say the $s$-abacus of $\lambda$ exhibits $(RL, +)$ symmetry if there is a bead in the $(i, j)$ position if and only if there is a bead in the $(s-1-i, j)$ position. [RL here refers to right-left.]

Lemma 16. The $s$-abacus of $\lambda$ exhibits $(RL, +)$ symmetry if and only if runner $i$ and runner $s-i-1$ have the same number of beads, and $\lambda(i) = \lambda(s-1-i)$ for $0 \leq i \leq s-1$.

Proof. Suppose the $s$-abacus of $\lambda$ exhibits $(RL, +)$ symmetry. Then each runner $i$ and $s-i-1$ must be identical. This means runners $i$ and $s-i-1$ have the same number of beads and $\lambda(i) = \lambda(s-1-i)$ for each $0 \leq i \leq s-1$. The proof in the other direction is clear. \qed

Theorem 19. $\lambda$ exhibits both $(UD, -)$ and $(RL, +)$ symmetry with respect to $s$ if and only if $q$ is even and the following three conditions hold for all $0 \leq i \leq s-1$:

1. $\lambda^0 = \emptyset$
2. $\lambda(i) = \lambda^*(i)$
3. $\lambda(i) = \lambda(s-1-i)$.

Proof. This follows from Lemma 7, Lemma 15, and Lemma 16. \qed

Example 20. The minimal 4-abacus of $\lambda = (8, 6, 6, 6, 6, 6, 6, 1, 1)$ exhibits $(UD, -)$ and $(RL, +)$ symmetry, but is neither a 3-core nor a 5-core. See Figure 5.

The following corollary is immediate.

Corollary 3. Let $s = 2k > 1$. The $s$-abacus of $\kappa_{s\pm1}$ exhibits $(UD, -)$ and $(RL, +)$ symmetry.

Corollary 4. If the $s$-abacus of $\lambda$ exhibits both $(UD, -)$ and $(RL, +)$ then $\lambda$ is self-conjugate and has empty $s$-core.

Proof. By Theorem 19, since $\lambda(i) = \lambda(s-1-i)$ and $\lambda(i) = \lambda^*(i)$, we have $\lambda(i) = \lambda^*(s-1-i)$. Since $\lambda^0 = \emptyset$, and by assumption $|X| = 0 \pmod{s}$, we have $\lambda = \lambda^*$ by Lemma 6. \qed

5. Simultaneous $(s-1, s, s+1)$-cores

5.1. An $s$-abacus Characterization of the Longest $(2k-1, 2k, 2k+1)$-core

A conjecture of Amdeberhan [5] on the size of a maximal $(s-1, s, s+1)$-core has recently been verified.
Theorem 21. (Yang-Zhong-Zhou, [24]; Xiong, [23]) The size of the largest $(s-1, s, s+1)$-core is

1. $k\binom{k+1}{3}$ if $s = 2k > 2$

2. $(k + 1)\binom{k+1}{3} + \binom{k+2}{3}$ if $s = 2k + 1 > 2$.

The result is proved in two different ways: Yang, Zhong and Zhou extend the ideas of Stanley and Zanello to examine a poset $P_{s-1,s,s+1}$ associated to $(s-1, s, s+1)$-cores; for Xiong it is a consequence of numerical properties of bead sets associated to $(s-1, s, s+1, s+2, \ldots, s+k)$-cores. Here we find a characterization of the longest $s$-abacus, that is, the one corresponding to the $(s-1, s, s+1)$-core with the most components, and show that it corresponds to a maximal $(s-1, s, s+1)$-core.

We say that an $s$-abacus $\alpha(s)$ is a sub-abacus of $\alpha(s)$ if they have the same number of runners and $(i, j) \in \alpha'(s)$ implies that $(i, j) \in \alpha(s)$. Let $\bar{\alpha}(s)$ be the sub-abacus of $\alpha(s)$ obtained by deleting any bead in $\alpha(s)$ which has a spacer directly below it on the same runner.

Lemma 17. The $s$-abacus $\bar{\alpha}(s)$ corresponds to an $s$-core partition.

Proof. This follows from Lemma 4, since, by construction, there is no bead in any runner with a spacer below it.

Lemma 18. Let $0 \leq i \leq k - 1$. The $s$-abacus $\bar{\alpha}(s)$ consists of consecutive beads in the rows $j = 0, 1, \ldots, i$ of the $i$ and $s - 1 - i$ runners.

Proof. This follows from the construction of $\alpha(s)$ in Definition 11, where the $i$ and $s - i - 1$ runners have beads in the rows $0, 1, \ldots, i$, followed by alternating spacer-bead sequences.

Example 22. Consider the 10-abacus $\bar{\alpha}(10)$ in Appendix C, Figure 21. Runners 0 and 9 have no beads, runners 1 and 8 have one bead, runners 2 and 7 have two beads and so on.

Lemma 19. Let $0 \leq j \leq k - 2$. Reading from left-to-right, row $j$ of the $s$-abacus $\bar{\alpha}(s)$ consists of $j+1$ spacers, followed by $s-2(j+1)$ beads, followed by $j+1$ spacers.

Proof. This follows from Lemma 18.

Example 23. Consider the 10-abacus $\bar{\alpha}(10)$ in Appendix C, Figure 21. Row 0 has a spacer followed by eight beads, followed by a spacer. Row 1 has two spacers followed by six beads, followed by two spacers, and so on.

Lemma 20. Let $j > 0$. If $(i, j) \in \bar{\alpha}(s)$, then $(i-1, j-1) \in \bar{\alpha}(s)$ and $(i+1, j-1) \in \bar{\alpha}(s)$. 

Proof. This is equivalent to saying that each bead in the second row of \( \alpha(s) \) or above has a bead one row below and one column to the right, and one row below to the left, which follows from Lemma 19.

Example 24. Consider the 10-abacus \( \bar{\alpha}(10) \) in Appendix C, Figure 21. The bead in position (2,1) (with bead-value 12) is flanked by beads in positions (1,0) and (3,0) (with bead-values 1 and 3 respectively).

Let \( \kappa_{s-1,s,s+1} \) be the \((s-1, s, s+1)\)-core with the largest number of components.

Lemma 21. The \( s \)-abacus \( \bar{\alpha}(s) \) corresponds to the \( \kappa_{s-1,s,s+1} \), that is, the one with the most components.

Proof. By Remark 1 of [6], the \( s \)-abacus of any \((s-1, s+1)\)-core partition must be a sub-abacus of \( \alpha(s) \). Then \( \bar{\alpha}(s) \) must be the \( s \)-abacus of \( \kappa_{s-1,s,s+1} \) since it is obtained by deleting any bead in \( \alpha(s) \) with a spacer immediately below it. It also must be the sub-abacus with the most beads, since including another bead would mean an \( s \)-hook is introduced.

Since \( \alpha(s) \) was a minimal bead set, so too is \( \bar{\alpha}(s) \); since each bead corresponds to a component, this means \( \bar{\alpha}(s) \) is the \((s-1, s, s+1)\)-core with the most components.

We denote the \((s-1, s, s+1)\)-core corresponding to \( \bar{\alpha}(s) \) by \( \kappa_{s-1,s,s+1} \).

Lemma 22. Each bead in row \( j \) of \( \bar{\alpha}(s) \) corresponds to a size \((j+1)^2\) component of \( \kappa_{s-1,s,s+1} \).

Proof. By Lemma 1, a bead \( x \) corresponds to a partition component whose size is the number of spacers less than \( x \). We use induction on \( j \). It is clear that each bead in the row 0 corresponds to a component of size 1 = \((0+1)^2\). Suppose it is true for \( j-1 \). Then, by the inductive hypothesis, there are \( j^2 \) spacers less than any bead in row \( j-1 \). By Lemma 19, the number of spacers between a bead in row \( j-1 \) and a bead in row \( j \) is \( j + (j+1) \). Since the number of spacers less than a bead in row \( j \) is \( j^2 + j + (j+1) = (j+1)^2 \) we are done.

The next corollary follows from Lemma 22. [See Theorem 2.5 in [16] for more details.]

Corollary 5. Let \( s = 2k > 2 \). Then \( \kappa_{s-1,s,s+1} = (((k-1)^2)^2, ((k-2)^2)^4, \ldots, 16^{2k-8}, 4^{2k-12}, 1^{2k-2}) \).

We are now in a position to prove that \( \kappa_{s-1,s,s+1} \) the longest \((s-1, s, s+1)\)-core is a maximal one.

Theorem 25. Let \( s = 2k > 2 \). Then \( \kappa_{s-1,s,s+1} \) is a maximal \((s-1, s, s+1)\)-core.
Proof. By Lemma 22, each bead in row \( j \) corresponds to a size \((j + 1)^2\) component of \( \kappa_{s-1,s,s+1} \). By Lemma 19, there are \((s - 2(j + 1))\) beads in row \( j \). Hence

\[
|\kappa_{s-1,s,s+1}| = \sum_{j=0}^{k-1} (s - 2(j + 1))(j + 1)^2.
\]

Since

\[
\sum_{j=0}^{k-1} (s - 2(j + 1))(j + 1)^2 = 2k \sum_{j=0}^{k-1} (j + 1)^2 - 2 \sum_{j=0}^{k-1} (j + 1)^3
= 2k \sum_{j=1}^{k} j^2 - 2 \sum_{j=1}^{k} j^3
= \frac{2k(k+1)(2k+1)}{6} - \frac{3k^2(k+1)^2}{6}
= \frac{k^2(k^2 - 1)}{6}
= k\left(\frac{k+1}{3}\right),
\]

we are done. \( \Box \)

5.2. Further Directions

Theorem 21 allows us to compare \( |\kappa_{s\pm 1}| \) with \( |\kappa_{s-1,s,s+1}| \) when \( s = 2k > 2 \).

**Proposition 2.** Let \( s = 2k > 2 \). Then \( |\kappa_{s\pm 1}| > |\kappa_{s-1,s,s+1}| \). In particular, \( |\kappa_{s\pm 1}| = 4|\kappa_{s-1,s,s+1}| \)

Proof. Since \( s \) is even, by Theorem 21(1) above \( |\kappa_{s-1,s,s+1}| = \frac{k^4-k^2}{24} \). However by Theorem 4, \( |\kappa_{s\pm 1}| = \frac{(s-1)^2(s+1)^2-1}{24} \). This simplifies to \( \frac{4(k^4-k^2)}{6} \). The result follows. \( \Box \)

**Corollary 6.** \( \kappa_{s-1,s+1} \) is never an \( s \)-core.

Corollary 6 also follows from Theorem 1: since \( \kappa_{s\pm 1} \) is expressed only in its \( s \)-quotient (its \( s \)-core is empty), and each 1-hook in the \( s \)-quotient corresponds to an \( s \)-hook of \( \kappa_{s\pm 1} \), the maximal \((s - 1, s + 1)\)-core is comprised completely of \( s \)-hooks.

Several questions arise from the analysis in this section. Firstly, is there interpretation of the factor of 4 that appears in Proposition 2, either in the geometry of the \( s \)-abacus or in the manipulation of Young diagrams? A cursory comparison between \( \bar{\alpha}(s) \) and \( \alpha(s) \) does not suggest an obvious one (compare Appendix A and Appendix C, for example). Secondly, Aggarwal, Yang-Zhong-Zhou and Xiong all note that when \( s = 2k > 2 \), there are two maximal \((s - 1, s, s + 1)\)-cores, and in particular,
$\kappa_{s-1, s+1}$ is not self-conjugate. What then is the size of maximal self-conjugate $(s - 1, s, s + 1)$-core in this case?

Finally, Yang-Zhong-Zhou spend several pages establishing that the longest $(s - 1, s, s + 1)$-core is of maximal size. Is there a shorter abacus proof that this is the case? If so, then our characterization of $\bar{\alpha}(s)$ could be employed to develop a new proof of Theorem 21. Aggarwal has commented that it is not known for general, distinct, $s, t, u$ when the longest $(s, t, u)$-core is a maximal one.

These questions are beyond the scope of this paper; we leave their investigation to other venues.

**Note:** since these results first appeared, a combinatorial explanation for the factor of 4 that appears in Proposition 2 has been found by the author and J. Sellers [16].

**Acknowledgements** This paper was conceived while visiting the University of Minnesota Duluth REU in July 2014; the author thanks Joe Gallian for the invitation and hospitality while there. The author thanks Christopher Hanusa for his valuable comments on the manuscript and his suggestions on diagrams. The author also thanks Amol Aggarwal for helpful conversations. Finally the author thanks the anonymous referee for their careful readings and thoughtful comments: questions raised by the referee lead to the inclusion of the new characterizations in Section 5.1, and some of the questions in Section 5.2.

**References**


[21] J. Vandehey, Containment in $(s,t)$-core partitions, Undergraduate Thesis (2008), University of Oregon.


APPENDIX A
The $s$-abaci $\alpha(s)$ of $\kappa_{s \pm 1}$

Figure 9: $s = 4$

4 5 6 7
0 1 2 3

Figure 10: $s = 6$

18 19 20 21 22 23
12 13 14 15 16 17
6 7 8 9 10 11
0 1 2 3 4 5

Figure 11: $s = 8$

40 41 42 43 44 45 46 47
32 33 34 35 36 37 38 39
24 25 26 27 28 29 30 31
16 17 18 19 20 21 22 23
8 9 10 11 12 13 14 15
0 1 2 3 4 5 6 7

Figure 12: $s = 10$

70 71 72 73 74 75 76 77 78 79
60 61 62 63 64 65 66 67 68 69
50 51 52 53 54 55 56 57 58 59
40 41 42 43 44 45 46 47 48 49
30 31 32 33 34 35 36 37 38 39
20 21 22 23 24 25 26 27 28 29
10 11 12 13 14 15 16 17 18 19
0 1 2 3 4 5 6 7 8 9
APPENDIX B

The $s$-quotients of $\kappa_{s\pm 1}$

Figure 13: 4-quotient of $\kappa_{3,5}$

\[
\begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ }
\end{array}
\]

Figure 14: 6-quotient of $\kappa_{5,7}$

\[
\begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ }
\end{array}
\]

Figure 15: 8-quotient of $\kappa_{7,9}$

\[
\begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ }
\end{array}
\]

Figure 16: 10-quotient of $\kappa_{9,11}$

\[
\begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ }
\end{array}
\]
APPENDIX C

The $s$-abaci $\bar{\alpha}(s)$ of $\kappa_{s-1,s,s+1}$

Figure 17: $s = 4$

```
4 5 6 7
0 1 2 3
```

Figure 18: $s = 6$

```
18 19 20 21 22 23
12 13 14 15 16 17
6 7 8 9 10 11
0 1 2 3 4 5
```

Figure 19: $s = 8$

```
40 41 42 43 44 45 46 47
32 33 34 35 36 37 38 39
24 25 26 27 28 29 30 31
16 17 18 19 20 21 22 23
8 9 10 11 12 13 14 15
0 1 2 3 4 5 6 7
```

Figure 20: $s = 10$

```
70 71 72 73 74 75 76 77 78 79
60 61 62 63 64 65 66 67 68 69
50 51 52 53 54 55 56 57 58 59
40 41 42 43 44 45 46 47 48 49
30 31 32 33 34 35 36 37 38 39
20 21 22 23 24 25 26 27 28 29
10 11 12 13 14 15 16 17 18 19
0 1 2 3 4 5 6 7 8 9
```