INTEGER SETS WITH IDENTICAL REPRESENTATION FUNCTIONS

Yong-Gao Chen
School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing, China
ygchen@njnu.edu.cn

Vsevolod F. Lev
Department of Mathematics, The University of Haifa at Oranim, Tivon, Israel
seva@math.haifa.ac.il

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Abstract
We present a versatile construction allowing one to obtain pairs of integer sets with infinite symmetric difference, infinite intersection, and identical representation functions.

Let \( \mathbb{N}_0 \) denote the set of all non-negative integers. To every subset \( A \subseteq \mathbb{N}_0 \) corresponds its representation function \( R_A \) defined by

\[
R_A(n) := |\{(a', a'') \in A \times A: n = a' + a'', a' < a''\}|;
\]

that is, \( R_A(n) \) is the number of unordered representations of the integer \( n \) as a sum of two distinct elements of \( A \).

Answering a question of Sárközy, Dombi [4] constructed sets \( A, B \subseteq \mathbb{N}_0 \) with infinite symmetric difference such that \( R_A = R_B \). The result of Dombi was further extended and developed in [3] (where a different representation function was considered) and [5] (a simple common proof of the results from [4] and [3] using generating functions); other related results can be found in [1, 2, 6, 8].

The two sets constructed by Dombi actually partition the ground set \( \mathbb{N}_0 \), which makes one wonder whether one can find \( A, B \subseteq \mathbb{N}_0 \) with \( R_A = R_B \) so that not only the symmetric difference of \( A \) and \( B \), but also their intersection is infinite. Tang and Yu [9] proved that if \( A \cup B = \mathbb{N}_0 \) and \( R_A(n) = R_B(n) \) for all sufficiently large integers \( n \), then at least one cannot have \( A \cap B = 4\mathbb{N}_0 \) (here and below \( k\mathbb{N}_0 \) denotes the dilate of the set \( \mathbb{N}_0 \) by the factor \( k \)). They further conjectured that, indeed, under the same assumptions, the intersection \( A \cap B \) cannot be an infinite

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arithmetic progression, unless \( A = B = \mathbb{N}_0 \). The main goal of this note is to resolve the conjecture of Tang and Yu in the negative by constructing an infinite family of pairs of sets \( A, B \subseteq \mathbb{N}_0 \) with \( R_A = R_B \) such that \( A \cup B = \mathbb{N}_0 \), while \( A \cap B \) is an infinite arithmetic progression properly contained in \( \mathbb{N}_0 \). Our method also allows one to easily construct sets \( A, B \subseteq \mathbb{N}_0 \) with \( R_A = R_B \) such that both their symmetric difference and intersection are infinite, while their union is arbitrarily sparse and the intersection is not an arithmetic progression.

For sets \( A, B \subseteq \mathbb{N}_0 \) and integer \( m \), let \( A - B := \{a - b: (a, b) \in A \times B\} \) and \( m + A := \{m + a: a \in A\} \).

The following basic lemma is in the heart of our construction.

**Lemma 1.** Suppose that \( A_0, B_0 \subseteq \mathbb{N}_0 \) satisfy \( R_{A_0} = R_{B_0} \), and that \( m \) is a non-negative integer with \( m \notin (A_0 - B_0) \cup (B_0 - A_0) \). Then, letting

\[
A_1 := A_0 \cup (m + B_0) \text{ and } B_1 := B_0 \cup (m + A_0),
\]

we have \( R_{A_1} = R_{B_1} \) and furthermore

i) \( A_1 \cup B_1 = (A_0 \cup B_0) \cup (m + A_0 \cup B_0) \);

ii) \( A_1 \cap B_1 \supseteq (A_0 \cap B_0) \cup (m + A_0 \cap B_0) \), the union being disjoint.

Moreover, if \( m \notin (A_0 - A_0) \cup (B_0 - B_0) \), then also in i) the union is disjoint, and in ii) the inclusion is in fact an equality. In particular, if \( A_0 \cup B_0 = [0, m - 1] \), then \( A_1 \cup B_1 = [0, 2m - 1] \), and if \( A_0 \) and \( B_0 \) indeed partition the interval \([0, m - 1]\), then \( A_1 \) and \( B_1 \) partition the interval \([0, 2m - 1]\).

**Proof.** Since the assumption \( m \notin A_0 - B_0 \) ensures that \( A_0 \) is disjoint from \( m + B_0 \), for any integer \( n \) we have

\[
R_{A_1}(n) = R_{A_0}(n) + R_{B_0}(n - 2m) + |\{(a_0, b_0) \in A_0 \times B_0: a_0 + b_0 = n - m\}|.
\]

Similarly,

\[
R_{B_1}(n) = R_{B_0}(n) + R_{A_0}(n - 2m) + |\{(a_0, b_0) \in A_0 \times B_0: a_0 + b_0 = n - m\}|,
\]

and in view of \( R_{A_0} = R_{B_0} \), this gives \( R_{A_1} = R_{B_1} \). The remaining assertions are straightforward to verify. \( \square \)

Given subsets \( A_0, B_0 \subseteq \mathbb{N}_0 \) and a sequence \((m_i)_{i \in \mathbb{N}_0}\) with \( m_i \in \mathbb{N}_0 \) for each \( i \in \mathbb{N}_0 \), define subsequently

\[
A_i := A_{i-1} \cup (m_{i-1} + B_{i-1}) \text{ and } B_i := B_{i-1} \cup (m_{i-1} + A_{i-1}), \quad i = 1, 2, \ldots \quad (1)
\]

and let

\[
A := \bigcup_{i \in \mathbb{N}_0} A_i, \quad B := \bigcup_{i \in \mathbb{N}_0} B_i. \quad (2)
\]
As an immediate corollary of Lemma 1, if $R_{A_0} = R_{B_0}$ and $m_i \notin (A_i - B_i) \cup (B_i - A_i)$ for each $i \in \mathbb{N}_0$, then $R_A = R_B$.

The special case $A_0 = \{0\}$, $B_0 = \{1\}$, $m_i = 2^{i+1}$ yields the partition of Dombi (which, we remark, was originally expressed in completely different terms). Below we analyze yet another special case obtained by fixing arbitrarily an integer $l \geq 1$ and choosing $A_0 := \{0\}$, $B_0 := \{1\}$, and

$$m_i :=\begin{cases} 2^{i+1}, & 0 \leq i \leq 2l - 2, \\ 2^{2l} - 1, & i = 2l - 1, \\ 2^{i+1} - 2^{i-2l}, & i \geq 2l. \end{cases} \quad (3)$$

We notice that $R_{A_0} = R_{B_0}$ in a trivial way (both functions are identically equal to 0), and that $A_0$ and $B_0$ partition the interval $[0, m_0 - 1]$. Applying Lemma 1 inductively $2l - 2$ times, we conclude that in fact for each $i \leq 2l - 2$, the sets $A_i$ and $B_i$ partition the interval $[0, 2m_{i-1} - 1] = [0, m_i - 1]$, and consequently $m_i \notin (A_i - B_i) \cup (B_i - A_i)$ and $m_i \notin (A_i - A_i) \cup (B_i - B_i)$. In particular, $A_{2l-2}$ and $B_{2l-2}$ partition $[0, m_{2l-2} - 1]$, and therefore $A_{2l-1}$ and $B_{2l-1}$ partition $[0, 2m_{2l-2} - 1] = [0, m_{2l-1}]$.

In addition, it is easily seen that $A_{2l-1}$ contains both 0 and $m_{2l-1}$, whence $m_{2l-1} \in A_{2l-1} - A_{2l-1}$, but $m_{2l-1} \notin B_{2l-1} - B_{2l-1}$ and $m_{2l-1} \notin (A_{2l-1} - B_{2l-1}) \cup (B_{2l-1} - A_{2l-1})$. From Lemma 1 i) it follows now that $A_{2l} \cup B_{2l} = [0, 2m_{2l-1}] = [0, m_{2l-1}]$, while

$$A_{2l} \cap B_{2l} = \left( A_{2l-1} \cap (m_{2l-1} + A_{2l-1}) \right) \cup \left( B_{2l-1} \cap (m_{2l-1} + B_{2l-1}) \right) = \{m_{2l-1}\}.$$

Applying again Lemma 1 we then conclude that for each $i \geq 2l$,

$$A_i \cup B_i = [0, m_i - 1]$$

(implying $m_i \notin (A_i - B_i) \cup (B_i - A_i) \cup (A_i - A_i) \cup (B_i - B_i)$) and

$$A_i \cap B_i = m_{2l-1} + \{0, m_{2l}, 2m_{2l}, \ldots, (2^{i-2l} - 1)m_{2l}\}.$$

As a result, with $A$ and $B$ defined by (2), we have $A \cup B = \mathbb{N}_0$ while the intersection of $A$ and $B$ is the infinite arithmetic progression $m_{2l-1} + m_{2l}\mathbb{N}_0$. Moreover, the condition $m_i \notin (A_i - B_i) \cup (B_i - A_i)$, which we have verified above to hold for each $i \geq 0$, results in $R_A = R_B$.

We thus have proved the following result.

**Theorem 1.** Let $l$ be a positive integer, and suppose that the sets $A, B \subseteq \mathbb{N}_0$ are obtained as in (1)-(2) starting from $A_0 = \{0\}$ and $B_0 = \{1\}$, with $(m_i)$ defined by (3). Then $R_A = R_B$, while $A \cup B = \mathbb{N}_0$ and $A \cap B = (2^{2l} - 1) + (2^{2l+1} - 1) \mathbb{N}_0$.

We notice that for any fixed integers $r \geq 2^{2l} - 1$ and $m \geq 2^{2l+1} - 1$, having (3) appropriately modified (namely, setting $m_i = 2^{i-2l}m$ for $i \geq 2l$) and translating $A$
and $B$, one can replace the progression $(2^{2^l} - 1) + (2^{2^l+1} - 1)\mathbb{N}_0$ in the statement
of Theorem 1 with the progression $r + m\mathbb{N}_0$; however, the relation $A \cup B = \mathbb{N}_0$ will
not hold true any longer unless $r = 2^{2^l} - 1$ and $m = 2^{2^l+1} - 1$. This suggests the
following question.

**Problem 1.** Given that $R_A = R_B$, $A \cup B = \mathbb{N}_0$, and $A \cap B = r + m\mathbb{N}_0$ with
integer $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2^l} - 1$,
$m = 2^{2^l+1} - 1$, and $A, B$ are as in Theorem 1?

The finite version of this question is as follows.

**Problem 2.** Given that $R_A = R_B$, $A \cup B = [0, m - 1]$, and $A \cap B = \{r\}$ with
integers $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2^l} - 1$,
$m = 2^{2^l+1} - 1$, $A = A_{2l}$, and $B = B_{2l}$, with $A_{2l}$ nd $B_{2l}$ as in the proof of Theorem 1?

We conclude our note with yet another natural problem.

**Problem 3.** Do there exist sets $A, B \subseteq \mathbb{N}_0$ with the infinite symmetric difference
and with $R_A = R_B$ which cannot be obtained by a repeated application of
Lemma 1?

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**References**


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