q-MULTIPARAMETER-BERNOULLI POLYNOMIALS AND
q-MULTIPARAMETER-CAUCHY POLYNOMIALS BY JACKSON’S
INTEGRALS

Takao Komatsu
School of Mathematics and Statistics, Wuhan University, Wuhan, China
komatsu@whu.edu.cn

László Szalay
Institute of Mathematics, University of West Hungary, Sopron, Hungary
and
Department of Mathematics and Informatics, University J. Selye, Komarno,
Slovakia
szalay.laszlo@emk.nyme.hu

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Abstract
We define q-multiparameter-Bernoulli polynomials and q-multiparameter-Cauchy polynomials by using Jackson’s integrals, which generalize the previously known numbers, including poly-Bernoulli $B_n^{(k)}$ and the poly-Cauchy numbers of the first kind $c_n^{(k)}$ and of the second kind $d_n^{(k)}$. We investigate their properties connected with multiparameter Stirling numbers which generalize the original Stirling numbers. We also give the relations between q-multiparameter-Bernoulli polynomials and q-
multiparameter-Cauchy polynomials.

1. Introduction

Let $n$ and $k$ be integers with $n \geq 0$, and let $L = (l_1, \ldots, l_k)$ be a $k$-tuple of real numbers with $\ell := l_1 \cdots l_k \neq 0$ and $A = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ be a $n$-tuple of real numbers. Let $q$ be a real number with $0 \leq q < 1$.

Jackson’s $q$-derivative with $0 < q < 1$ (see e.g., [1, (10.2.3)], [12]) is defined by

$$D_q f = \frac{d_q f}{d_q x} = \frac{f(x) - f(qx)}{(1 - q)x}$$
and Jackson’s $q$-integral ([1, (10.1.3)], [12]) is defined by
\[
\int_0^x f(t)d_qt = (1-q)x \sum_{n=0}^{\infty} f(q^n x)q^n.
\]
The Jackson integral gives a unique $q$-antiderivative within a certain class of functions. In particular, when $f(x) = x^m$ for some nonnegative integer $m$, then
\[
D_qf = \frac{x^m - q^m x^m}{(1-q)x} = [m]_q x^{m-1}
\]
and
\[
\int_0^x t^m d_qt = (1-q)x \sum_{n=0}^{\infty} q^m t^n q^n = (1-q)x^{m+1} \sum_{n=0}^{\infty} q^{n(m+1)} = x^{m+1} \frac{[m+1]_q}{[m+1]_q}.
\]
Here,
\[
[x]_q = \frac{1-q^x}{1-q}
\]
is the $q$-number with $[0]_q = 0$ (see e.g. [1, (10.2.3)], [12]). Note that $\lim_{q \to 1} [x]_q = x$.

Define poly-Bernoulli polynomials $B_{n,\rho,q}(z)$ with a parameter $\rho$ by
\[
\frac{\rho}{1-e^{-\rho t}} \text{Li}_{k,q} \left( \frac{1-e^{-\rho t}}{\rho} \right) e^{-tz} = \sum_{n=0}^{\infty} B_{n,\rho,q}^{(k)}(z) \frac{t^n}{n!}, \tag{1}
\]
where $\text{Li}_{k,q}(z)$ is the $q$-polylogarithm function (see [16]) defined by
\[
\text{Li}_{k,q}(z) = \sum_{n=1}^{\infty} \frac{z^n}{[n]_q^k}.
\]
Notice that
\[
\lim_{q \to 1} B_{n,\rho,q}^{(k)}(z) = B_{n,\rho}^{(k)}(z),
\]
which is the poly-Bernoulli polynomial with a $\rho$ parameter (see [6]), and
\[
\lim_{q \to 1} \text{Li}_{k,q}(z) = \text{Li}_{k}(z),
\]
which is the ordinary polylogarithm function, defined by
\[
\text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}. \tag{2}
\]
In addition, when \( z = 0 \), \( B_{n,\rho}^{(k)}(0) = B_{n,\rho}^{(k)} \) is the poly-Bernoulli number with a \( \rho \) parameter. When \( z = 0 \) and \( \rho = 1 \), \( B_{n,1}^{(k)}(0) = B_{n}^{(k)} \) is the poly-Bernoulli number (see [15]) defined by
\[
\text{Li}_k(1 - e^{-t}) = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}.
\] (3)

The poly-Bernoulli numbers are expressed as special values at negative arguments of certain combinations of multiple zeta values. The poly-Bernoulli numbers can be expressed in terms of the Stirling numbers of the second kind.
\[
B_n^{(k)} = \sum_{m=0}^{n} \frac{(-1)^{n-m}m!S_2(n,m)}{(m+1)^k} \quad (n \geq 0, \ k \geq 1)
\] ([15, Theorem 1]), where \( S_2(n,m) \) is the Stirling number of the second kind, see [7], determined by the falling factorial:
\[
x^n = \sum_{m=0}^{n} S_2(n,m)x(x-1)\cdots(x-m+1).
\]

The poly-Bernoulli numbers are extended to the poly-Bernoulli polynomials (see [3, 8]) and to the special multi-poly-Bernoulli numbers (see [11]). The Bernoulli polynomials occur in the study of many special functions and in particular the Riemann zeta function and the Hurwitz zeta function. They are an Appell sequence, i.e., a Sheffer sequence for the ordinary derivative operator.

Define the \( q \)-multiparameter-poly-Cauchy polynomials of the first kind \( c_{n,L,A,q}^{(k)}(z) \) by
\[
c_{n,L,A,q}^{(k)}(z) = \int_{0}^{l_1} \cdots \int_{0}^{l_k} (x_1 \cdots x_k - \alpha_0 - z) \cdots (x_1 \cdots x_k - \alpha_{n-1} - z) d_q x_1 \cdots d_q x_k.
\] (4)

Notice that
\[
\lim_{q \to 1} c_{n,L,A,q}^{(k)}(z) = c_{n,L,A}^{(k)}(z),
\]
which are the multiparameter-poly-Cauchy polynomials of the first kind. The idea of dealing with multiparameters \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \) instead of 0, 1, \ldots, \( n-1 \) has already been considered in [25]. Namely, If \( l_1 = \cdots = l_k = 1 \) and \( z = 0 \), the number \( c_{n,(1,\ldots,1),A}^{(k)} = c_{n,A}^{(k)} \) has been studied to prove the convexity. It has been proven that \( c_{n,A}^{(k)} \) is log-convex, satisfying \( (c_{n,A}^{(k)})^2 - c_{n-1,A}^{(k)} c_{n+1,A}^{(k)} \leq 0 \).

In addition, if \( \alpha_i = i\rho \) (\( i = 0, 1, \ldots, n-1 \)), then the number \( c_{n,A}^{(k)} \) is reduced to the poly-Cauchy numbers of the first kind with a parameter \( \rho \) (see [19]). Furthermore, if \( \rho = 1 \), then the number \( c_{n,A}^{(k)} \) is reduced to the poly-Cauchy number \( c_n^{(k)} \) (see [18]). If \( k = 1 \), then \( c_n^{(1)} = c_n \) is the classical Cauchy number (see [7, 27]). The
number \( c_n/n! \) is sometimes referred to as the Bernoulli number of the second kind (see [4, 13, 28]).

The poly-Cauchy numbers have been considered as analogues of the poly-Bernoulli numbers \( B_n^{(k)} \). The poly-Cauchy numbers of the first kind, \( c_n^{(k)} \), can be expressed in terms of the Stirling numbers of the first kind:

\[
c_n^{(k)} = \sum_{m=0}^{n} \frac{(-1)^{n-m} S_1(n, m)}{(m+1)^k} \quad (n \geq 0, \ k \geq 1)
\]

([18, Theorem 1]), where \( S_1(n, m) \) is the (unsigned) Stirling number of the first kind (see [7]), determined by the rising factorial:

\[
x(x+1) \cdots (x+n-1) = \sum_{m=0}^{n} S_1(n, m)x^m.
\]  

(5)

Similarly, define the \( q \)-multiparameter-poly-Cauchy polynomials of the second kind \( \tilde{c}_n^{(k)}(z) \) by

\[
\tilde{c}_n^{(k)}(z) = \int_{0}^{t_1} \cdots \int_{0}^{t_k} (-x_1 \cdots x_k - \alpha_0 + z) \cdots (-x_1 \cdots x_k - \alpha_{n-1} + z) d_0 x_1 \cdots d_k x_k.
\]  

(6)

If \( q \to 1, l_1 = \cdots = l_k = 1, \alpha_i = i \rho (i = 0, 1, \ldots, n-1) \) and \( z = 0 \), the number \( \tilde{c}_n^{(k)} \) is reduced to the poly-Cauchy numbers of the second kind with a parameter \( \rho \) (see [19]). Furthermore, if \( \rho = 1 \), then the number \( \tilde{c}_n^{(k)} \) is reduced to the poly-Cauchy numbers of the second kind \( c_n^{(k)} \) (see [18]). If \( k = 1 \), then \( c_n^{(1)} = \tilde{c}_n^{(1)} \) is the classical Cauchy number (see [7, 27]). The poly-Cauchy numbers of the second kind \( \tilde{c}_n^{(k)} \) can be expressed in terms of the Stirling numbers of the first kind by

\[
\tilde{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{S_1(n, m)}{(m+1)^k} \quad (n \geq 0, \ k \geq 1)
\]

([18, Theorem 4]). The generating function of the poly-Cauchy numbers of the second kind \( \tilde{c}_n^{(k)} \) is given by

\[
Lif_k(-\ln(1+t)) = \sum_{n=0}^{\infty} \frac{\tilde{c}_n^{(k)} t^n}{n!}
\]  

(7)

([18, Theorem 5]).

The poly-Cauchy numbers (of the both kinds) are extended to the poly-Cauchy polynomials (see [14]), and to the poly-Cauchy numbers with a \( q \) parameter (see [19]). The corresponding poly-Bernoulli numbers with a \( q \) parameter can be obtained in [6]. A different direction of generalizations of Cauchy numbers is about
hypergeometric Cauchy numbers (see [21]). Arithmetical and combinatorial properties including sums of products have been studied (see [20, 23, 24]).

Various kinds of $q$-analogues or extensions have been studied. In [17], as generalizations of the poly-Cauchy numbers of the first kind $c_n^{(k)}$ and of the second kind $\tilde{c}_n^{(k)}$, by using Jackson’s $q$-integrals, $q$-analogues or extensions of the poly-Cauchy numbers of the first kind $c_{n,q}^{(k)}$ and of the second kind $\tilde{c}_{n,q}^{(k)}$ are introduced, and their properties are investigated. In [22], by using Jackson’s $q$-integrals, the concept about $q$-analogues or extensions of the poly-Bernoulli polynomials $B_{n,q}^{(k)}(z)$ with a parameter were also introduced.

In this paper, by using Jackson’s $q$-integrals, as essential generalizations of the previously known numbers and polynomials, including poly-Bernoulli numbers $B_n^{(k)}$, the poly-Cauchy numbers of the first kind $c_n^{(k)}$ and of the second kind $\tilde{c}_n^{(k)}$, we introduce the concept of $q$-analogues or extensions of the poly-Bernoulli polynomials $B_{n,p,q}^{(k)}(z)$ with a parameter, and the poly-Cauchy polynomials of the first kind $c_{n,q}^{(k)}$ and of the second kind $\tilde{c}_{n,q}^{(k)}$ with a parameter. We investigate their properties connected with the usual Stirling numbers and the weighted Stirling numbers. We also give the relations between generalized poly-Bernoulli polynomials and two kinds of generalized poly-Cauchy polynomials.

2. $q$-multiparameter-Cauchy Polynomials

For an $n$-tuple $A = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ of real numbers, define multiparameter Stirling numbers of the first kind $S_1(n, m, A)$ and of the second kind $S_2(n, m, A)$ by

$$ (t - \alpha_0)(t - \alpha_1) \cdots (t - \alpha_{n-1}) = \sum_{m=0}^{n} S_1(n, m, A)t^m $$

and

$$ \sum_{m=0}^{n} S_2(n, m, A)(t - \alpha_0)(t - \alpha_1) \cdots (t - \alpha_{m-1}) = t^n, $$

respectively (cf. [7, 9, 26]). If $\alpha_i = i\rho \ (i = 0, 1, \ldots, n - 1)$, then

$$ S_1(n, m, (0, \rho, \ldots, (n - 1)\rho)) = (-\rho)^{n-m}S_1(n, m), $$

$$ S_2(n, m, (0, \rho, \ldots, (n - 1)\rho)) = \rho^{n-m}S_2(n, m), $$

where $S_1(n, m)$ and $S_2(n, m)$ are the (unsigned) Stirling numbers of the first kind and the Stirling numbers of the second kind, respectively.

The $q$-multiparameter-poly-Cauchy polynomials of the first kind can be expressed explicitly in terms of the multiparameter Stirling numbers of the first kind.
Theorem 1. For all integers \( n \) and \( k \) with \( n \geq 0 \) and a real number \( q \) with \( 0 < q < 1 \), we have

\[
c_n(k, L, A, q) = \sum_{m=0}^{\infty} S_1(n, m, A) \sum_{i=0}^{m} \binom{m}{i} \left( \frac{(-z)^i q^{m-i+1}}{m-i+1} \right). \]

Proof. By definitions of (4) and (8), we have

\[
c_n(k, L, A, q) = \int_0^{x_1} \cdots \int_0^{x_k} \sum_{m=0}^{\infty} S_1(n, m, A) (x_1 \cdots x_k - z)^m d_q x_1 \cdots d_q x_k
\]

\[
= \sum_{m=0}^{\infty} S_1(n, m, A) \sum_{i=0}^{m} \binom{m}{i} \left( \frac{(-z)^i}{i+1} \right) x_1^i \cdots x_k^i d_q x_1 \cdots d_q x_k
\]

\[
= \sum_{m=0}^{\infty} S_1(n, m, A) \sum_{i=0}^{m} \binom{m}{i} \left( \frac{(-z)^i}{m-i+1} \right) q^{m-i+1}. \]

\[
\square
\]

If \( z = 0 \), then we have the expression of the \( q \)-multiparameter-poly-Cauchy numbers of the first kind.

Corollary 1. For all integers \( n \) and \( k \) with \( n \geq 0 \) and a real number \( q \) with \( 0 < q < 1 \), we have

\[
c_n(k, L, A, q) = \sum_{m=0}^{\infty} S_1(n, m, A) \frac{q^{m+1}}{m+1}. \]

Similarly, the \( q \)-multiparameter-poly-Cauchy polynomials of the second kind can be expressed explicitly in terms of the multiparameter Stirling numbers of the first kind. The proof is similar to that of Theorem 1 and is omitted.

Theorem 2. For all integers \( n \) and \( k \) with \( n \geq 0 \) and a real number \( q \) with \( 0 < q < 1 \), we have

\[
\hat{c}_n(k, L, A, q) = \sum_{m=0}^{\infty} \left( -1 \right)^m S_1(n, m, A) \sum_{i=0}^{m} \binom{m}{i} \left( \frac{(-z)^i q^{m-i+1}}{m-i+1} \right). \]

If \( z = 0 \), then we have the expression of the \( q \)-multiparameter-poly-Cauchy numbers of the second kind.
Corollary 2. For all integers $n$ and $k$ with $n \geq 0$ and a real number $q$ with $0 < q < 1$, we have
\[
\tilde{c}^{(k)}_{n,L,A,q} = \sum_{m=0}^{n} \frac{(-1)^m S_1(n, m, A)q^m}{[m+1]_q^k}.
\]

There are simple relations between two kinds of $q$-multiparameter-poly-Cauchy polynomials.

Theorem 3. For all integers $n$ and $k$ with $n \geq 1$ and a real number $q$ with $0 < q < 1$, we have
\[
(-1)^n c^{(k)}_{n,L,A,q}(z) = \tilde{c}^{(k)}_{n,L,A,q}(z), \tag{10}
\]
\[
(-1)^n c^{(k)}_{n,L,A,q}(z) = c^{(k)}_{n,L,-A,q}(z), \tag{11}
\]
where $-A = (-\alpha_0, -\alpha_1, \ldots, -\alpha_{n-1})$.

Proof. We shall prove identity (11). The identity (10) is proven similarly and omitted. By the definition of $\tilde{c}^{(k)}_{n,L,A,q}(z)$, we see that
\[
(-1)^n c^{(k)}_{n,L,A,q}(z) = c^{(k)}_{n,L,-A,q}(z).
\]
\[
\Box
\]

3. $q$-multiparameter-poly-Bernoulli Polynomials

Define the $q$-multiparameter-poly-Bernoulli polynomials $B^{(k)}_{n,L,A,q}(z)$ by
\[
B^{(k)}_{n,L,A,q}(z) = \sum_{m=0}^{n} S_2(n, m, A) m! \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i q^m-i+1}{[m-i+1]_q^k}. \tag{12}
\]

This is a generalization of poly-Bernoulli polynomials $B^{(k)}_{n}(z)$, defined in [24]. If $q \to 1$, $l_1 = \cdots = l_k = 1$ and $\alpha_i = i$ ($i = 0, 1, \ldots, n-1$), then the polynomial $B^{(k)}_{n,L,A,q}(z)$ are reduced to the polynomial $B^{(k)}_{n}(z)$ in [24].
By putting $z = 0$ in (12), the $q$-multiparameter-poly-Bernoulli numbers $B^{(k)}_{n,L,A,q}$ are given by

$$B^{(k)}_{n,L,A,q} = \sum_{m=0}^{n} \frac{S_2(n, m, A) m! q^{m+1}}{[m+1]_q^k}. \quad (13)$$

Since the orthogonality relations

$$\sum_{k=i}^{n} S_1(n, k, A) S_2(k, i, A) = \sum_{k=i}^{n} S_1(k, i, A) S_2(n, k, A) = \delta_{n,i}, \quad (14)$$

where $\delta_{n,i}$ is the Kronecker’s delta, we obtain the inverse relation

$$f_n = \sum_{m=0}^{n} S_1(n, m, A) g_m \iff g_n = \sum_{m=0}^{n} S_2(n, m, A) f_m. \quad (15)$$

**Theorem 4.** For $q$-multiparameter-poly-Bernoulli and $q$-multiparameter-poly-Cauchy polynomials, we have

$$\sum_{m=0}^{n} S_1(n, m, A) B^{(k)}_{m,L,A,q}(z) = n! \sum_{i=0}^{n} \binom{n}{i} \frac{(-z)^i q^{n-i+1}}{[n-i+1]_q^k}, \quad (16)$$

$$\sum_{m=0}^{n} S_2(n, m, A) c^{(k)}_{m,L,A,q}(z) = \sum_{i=0}^{n} \binom{n}{i} \frac{(-z)^i q^{n-i+1}}{[n-i+1]_q^k}, \quad (17)$$

$$\sum_{m=0}^{n} S_2(n, m, A) z^{(k)}_{m,L,A,q}(z) = (-1)^n \sum_{i=0}^{n} \binom{n}{i} \frac{(-z)^i q^{n-i+1}}{[n-i+1]_q^k}. \quad (18)$$

**Remark.** If $q \to 1$ and $\alpha_i = i \rho$ ($i = 0, 1, \ldots, n-1$), then Theorem 4 is reduced to Theorem 3.2 in [6].

**Proof.** By (12), applying (15) with

$$f_m = m! \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i q^{m-i+1}}{[m-i+1]_q^k} \quad \text{and} \quad g_n = B^{(k)}_{n,L,A,q}(z),$$

we get the identity (16). Similarly, by Theorem 1 and Theorem 2 we have the identities (17) and (18), respectively.

If we put $z = 0$ in Theorem 4, we have the identities for appropriate numbers.

**Corollary 3.** For $q$-multiparameter-poly-Bernoulli and $q$-multiparameter-poly-Cauchy
numbers, we have
\[
\sum_{m=0}^{n} S_1(n, m, A) B_{m,L,A,q}^{(k)} = \frac{n! q^{n+1}}{[n+1]_q}, \tag{19}
\]
\[
\sum_{m=0}^{n} S_2(n, m, A) c_{m,L,A,q}^{(k)} = \frac{\ell^n q^{n+1}}{[n+1]_q}, \tag{20}
\]
\[
\sum_{m=0}^{n} S_2(n, m, A) \tilde{c}_{m,L,A,q}^{(k)} = (-1)^n \frac{\ell^n q^{n+1}}{[n+1]_q}. \tag{21}
\]

4. Several Relations of $q$-poly-Bernoulli Polynomials and $q$-poly-Cauchy Polynomials

**Theorem 5.** For any $z$ we have
\[
B_{n,L,A,q}^{(k)}(z) = \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} m! S_2(n, m, A) S_2(m, \mu, A) c_{\mu,L,A,q}^{(k)}(z),
\]
\[
B_{n,L,A,q}^{(k)}(z) = \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} (-1)^m m! S_2(n, m, A) S_2(m, \mu, A) \tilde{c}_{\mu,L,A,q}^{(k)}(z),
\]
\[
c_{n,L,A,q}^{(k)}(z) = \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} \frac{1}{m!} S_1(n, m, A) S_1(m, \mu, A) B_{\mu,L,A,q}^{(k)}(z),
\]
\[
\tilde{c}_{n,L,A,q}^{(k)}(z) = \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} \frac{(-1)^m}{m!} S_1(n, m, A) S_1(m, \mu, A) B_{\mu,L,A,q}^{(k)}(z).
\]

**Remark.** If $\rho = 1$ and $q \to 1$ and $\alpha_i = i \rho$ ($i = 0, 1, \ldots, n - 1$), then Theorem 5 is reduced to Theorem 4.1 in [24]. A different generalization without Jackson’s integrals is discussed in [23].

**Proof.** We shall prove the first and the fourth identities. The other two are proven similarly and omitted. By (17) in Theorem 4 and (12), we have
\[
B_{n,L,A,q}^{(k)}(z) = \sum_{m=0}^{n} S_2(n, m, A) m! \sum_{\mu=0}^{m} S_2(m, \mu, A) c_{\mu,L,A,q}^{(k)}(z)
\]
\[
= \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} m! S_2(n, m, A) S_2(m, \mu, A) c_{\mu,L,A,q}^{(k)}(z).
\]
By (16) in Theorem 4 and Theorem 2, we have

\[
C_{n,L,A,q}^{(k)}(z) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} S_1(n, m, A) \sum_{\mu=0}^{m} S_1(m, \mu, A) B_{\mu,L,A,q}^{(k)}(z) \\
= \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} \frac{(-1)^m}{m!} S_1(n, m, A) S_1(m, \mu, A) B_{\mu,L,A,q}^{(k)}(z).
\]

\[\square\]

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