Abstract

We describe a new identity involving sums of powers of Fibonacci numbers and use this identity to prove that a certain family of combinatorial sequences converges, pointwise, to the Fibonacci sequence.

1. Introduction

We let $F$ represent the Fibonacci sequence where $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, and $F_{-n} = (-1)^n F_n$ for $n \in \mathbb{N}$. We then have $F_n = F_{n-1} + F_{n-2}$ for all $n \in \mathbb{Z}$. Our first main result is the following identity.

**Theorem 1.** For all $m \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\sum_{i=1}^{k+1} \left( F_{m-1} + (-1)^{k+1-i} \cdot F_{m-k+i-3} \right) \cdot \left( \frac{F_{m+1}}{F_m} \right)^i = F_{m+1} \cdot (F_{k+3} - 1).$$

If we clear denominators, the identity becomes

$$\sum_{i=1}^{k+1} \left( F_{m-1} + (-1)^{k+1-i} \cdot F_{m-k+i-3} \right) \cdot (F_{m+1}^{i} F_{m}^{k+1-i}) = F_{m+1}^{k+1} F_{m+1} \cdot (F_{k+3} - 1).$$

We could not find a similar or related identity in the literature, so this appears to be new. The closest identity we could find is the amazing four-parameter identity

$$F_{m}^{k} F_{n} = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h} (-1)^{h} F_{r}^{h} F_{r+m}^{k-h} F_{n+kr+hm}.$$
which can be used to produce many interesting known identities (see [5]).

We discovered the identity in Theorem 1 while studying rational base representations of natural numbers (see [1], [8], [3], [4], [6] or [2] for instance), which explains why the identity involves powers of \( \frac{F_{m+1}}{F_m} \). While these representations are quite complex from a language point of view, there is an elementary construction of an edge-labeled, infinite, rooted tree whose edge labels give the rational base representation of the integer associated to each vertex (see [1], [7] or [2]). It turns out that when using the rational base \( \frac{F_{m+1}}{F_m} \), the number of nodes lying distance \( n \) from the root in the associated tree is given by the sequence \( A^m \) with \( A^m_1 = 1 \) and

\[
A^m_{n+1} = \left[ \frac{F_{m+1}}{F_m} - \frac{F_m}{F_{m-1}} \cdot \sum_{i=1}^{n} A^m_i \right] = \left[ \frac{F_{m-1}}{F_m} \cdot \sum_{i=1}^{n} A^m_i \right]
\]

where \([x]\) represents the least integer larger than \( x \) (see [1] or [2]).

Interestingly, as \( m \) gets larger, the family of sequences \( \{A^m\} \) converges pointwise to the Fibonacci sequence \( F \). More precisely, we have the following theorem.

**Theorem 2.** Let \( \{A^m \mid m \geq 1\} \) be the family of sequences defined in (1). For every \( n \in \mathbb{N} \) with \( n \geq 1 \), there exists \( M \in \mathbb{N} \) such that \( A^m_n = F_n \) for all \( m \geq M \).

Thus, we have produced a family of sequences (with combinatorial interest) that can match the Fibonacci sequence for as many terms as we wish. Figure 1 shows the first 15 terms of the sequences \( A^m \) where \( m \in \{1, \ldots, 10\} \). The numbers in blue represent coincidence with \( F \). Note that \( A^{10} \) matches the Fibonacci sequence up to \( n = 15 \) (in fact \( n = 19 \) is the first index with \( A^{10}_n \neq F_n \)).

We also note that since \( \frac{F_{m-1}}{F_m} \rightarrow \frac{1}{\phi} \) as \( m \rightarrow \infty \) (where \( \phi \) represents the golden

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Figure 1: The first 15 terms of the sequences \( A^m \) for \( m \in \{1, \ldots, 10\} \). For instance, see A000007, A011782, and A073941 in [9].
ratio, \( \phi = \frac{1 + \sqrt{5}}{2} \), Theorem 2 implies that

\[
F_{n+1} = \left\lfloor \frac{1}{\phi} \sum_{i=1}^{n} F_i \right\rfloor
\]

with \( F_0 = 0 \) and \( F_1 = 1 \). While we could not find a citation for this formula, it must be known as it follows from well known facts. We know that \( F_{n+2} = \text{round}(\phi \cdot F_{n+1}) \) so that \( \frac{1}{\phi} F_{n+2} - \frac{1}{\phi^2} < F_{n+1} < \frac{1}{\phi} F_{n+2} + \frac{1}{\phi^2} \), which implies \( \frac{1}{\phi} F_{n+2} < F_{n+1} + \frac{1}{\phi} \) and \( F_{n+1} - \frac{1}{\phi} < \frac{1}{\phi} F_{n+2} \). Thus

\[
F_{n+1} - 1 < F_{n+1} - \frac{3}{2\phi} < \frac{1}{\phi} F_{n+2} - \frac{1}{\phi} F_{n+1} < \frac{1}{\phi} F_{n+2} < F_{n+1}
\]

so that

\[
\left\lfloor \frac{1}{\phi} \sum_{i=1}^{n} F_i \right\rfloor = \left\lfloor \frac{1}{\phi} (F_{n+2} - 1) \right\rfloor = F_{n+1}
\]

where the first equality is the well known formula for the sum of the first \( n \) Fibonacci numbers.

This note is organized as follows. In Section 2, we prove Theorem 1 using elementary techniques. In Section 3, we introduce the terminology of \( \mathbb{F}_2 \)-representations of natural numbers and state results from [1] in order to prove Theorem 2.

2. Proof of Theorem 1

To prove Theorem 1, we let \( m \in \mathbb{Z} \) and use induction on \( k \). For ease of notation, we define \( y_m := F_{m+1} F_{m}^{-1} \). We can check that the identity holds for \( k = 0 \) and \( k = 1 \). Indeed, we have (since \( F_3 - 1 = 2 - 1 = 1 \)):

\[
\sum_{i=1}^{1} (F_{m-1} + (-1)^{i-1} \cdot F_{m+i-3}) \cdot y_m^i = (F_{m-1} + F_{m-2}) y_m = F_{m+1} \cdot (F_3 - 1),
\]

and

\[
\sum_{i=1}^{2} (F_{m-1} + (-1)^{2i-1} \cdot F_{m+i-4}) \cdot y_m^i = (F_{m-1} - F_{m-3}) y_m + (F_{m-1} + F_{m-2}) y_m^2
\]

\[
= F_{m-2} \cdot y_m + F_m \cdot y_m^2
\]

\[
= F_{m+1} (F_{m-2} + F_{m+1})
\]

\[
= \frac{F_{m+1} (F_m - F_{m-1} + F_{m} + F_{m-1})}{F_m}
\]

\[
= F_{m+1} \cdot 2 = F_{m+1} \cdot (F_4 - 1).
\]

Now, let \( k \in \mathbb{N} \) with \( k \geq 1 \) and assume that the identity holds for \( j \in \{k-1, k\} \).
Notice that
\[ F_{m+1}(F_{k+4} - 1) = F_{m+1} + \frac{F_{m+1}(F_{k+3} - 1)}{A} + \frac{F_{m+1}(F_{k+2} - 1)}{B}. \]

Applying the inductive hypothesis to the quantities \(A\) and \(B\) in the previous equality yields
\[ A = \sum_{i=1}^{k+1} (F_{m-1} + (-1)^{k+1-i}F_{m-k+i-3})y_m^i \]
\[ B = \sum_{i=1}^{k} (F_{m-1} + (-1)^{k-i}F_{m-k+i-2})y_m^i \]
so that
\[ A + B = (F_{m-1} + F_{m-2}) \cdot y_m^{k+1} + \sum_{i=1}^{k} (2F_{m-1} + (-1)^{k-i}(F_{m-k+i-2} - F_{m-k+i-3}))y_m^i. \]

Rearranging sums and applying the Fibonacci identity leaves us with
\[ A + B = F_{m-2}y_m^{k+1} + F_{m-1}y_m^{k+1} + \sum_{i=1}^{k} F_{m-1}y_m^i + \sum_{i=1}^{k} (F_{m-1} + (-1)^{k-i}F_{m-k+i-4})y_m^i. \]

In the expression above, since \(F_{m+1} - F_m = F_{m-1}\), we know that
\[ C = F_{m-1} \sum_{i=1}^{k+1} y_m^i = C = F_{m-1} \cdot \left( \frac{y_m^{k+2} - 1}{y_m - 1} \right) = F_{m-1} \cdot \left( \frac{F_m y_m^{k+2} - F_m}{F_{m+1} - F_m} - 1 \right) \]
\[ = F_{m-1} \cdot \left( \frac{F_m y_m^{k+2} - F_m}{F_{m+1} - F_m} \right). \]

Therefore, we have
\[ F_{m+1}(F_{k+4} - 1) = F_{m+1} + F_{m-2}y_m^{k+1} + F_m y_m^{k+2} - F_m - F_{m-1} + D \]
\[ = F_{m-2}y_m^{k+1} + F_m y_m^{k+2} + D \]
since \(F_{m+1} - F_m - F_{m-1} = 0\). Next, \(F_{m-2} = F_{m-1} - F_{m-3}\) and \(F_m = F_{m-1} + F_{m-2}\), so that
\[ F_{m+1}(F_{k+4} - 1) = (F_{m-1} - F_{m-3})y_m^{k+1} + (F_{m-1} + F_{m-2})y_m^{k+2} + D \]
\[ = \sum_{i=1}^{k+2} (F_{m-1} + (-1)^{k-i}F_{m-k+i-4})y_m^i \]
\[ = \sum_{i=1}^{k+2} (F_{m-1} + (-1)^{k+2-i}F_{m-(k+1)+i-3})y_m^i, \]
as required. \(\square\)
3. $\mathbb{E}_q$-representations

For this section, we fix $p, q \in \mathbb{N}$ such that $p > q \geq 1$ and $\gcd(p, q) = 1$. For any $n \in \mathbb{N}$, we say $(n_0, n_1, \ldots, n_k)_q$ is a $\mathbb{E}_q$-representation for $n$ if $0 \leq n_i < p$ for all $i$ and $n = \sum_{i=0}^{k} n_i \left(\frac{p}{q}\right)^i$; in this case we write $n = (n_0, n_1, \ldots, n_k)_q$. We note that, unlike base-$b$ representations (with $b > 1$ an integer), not every string of digits, $(d_0, d_1, \ldots, d_k)_q$, yields a natural number. However, it is known from [1] (and earlier, see A024629 in [9] for instance) that every natural number $n$ has a unique $\mathbb{E}_q$-representation. Hence we can define $\text{len}_q(n) = k+1$ when $n = (n_0, n_1, \ldots, n_k)_q$. If the length of $n+1$ is larger than the length of $n$, i.e., $n \in \mathbb{N}$ satisfies $\text{len}_q(n+1) - \text{len}_q(n) = 1$, we say $n+1$ is new-length element (or nl-element for short).

Many properties of $\mathbb{E}_q$-representations (and related representations) are studied in [1] and [3], where the authors define an infinite, rooted tree, called $I_{p/q}$ that describes the $\mathbb{E}_q$-representations. A combinatorial construction of this tree is also given in [2] or [7]. In that tree, the nl-elements correspond to the nodes with the least label of any fixed distance from the root; these lie on the left branch of the tree when drawn as in [1].

To prove Theorem 2, we need the following results about $\mathbb{E}_q$-representations. We omit the proofs as these may be found in, or are straightforward consequences of, Proposition 21 and Corollary 23 in [1], though our terminology differs.

**Lemma 1.** Let $n$ be a natural number with $n = (n_0, n_1, \ldots, n_k)_q$. Then $n$ is an nl-element if and only if $n = 1$ or $n_0 = 0$, $0 \leq n_i < q$ for $1 \leq i \leq k-1$ and $n_k = q$.

Next, let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined by $g(n) = p \left\lfloor \frac{n}{q} \right\rfloor$.

**Proposition 1.** The sequence $(K_1, K_2, \ldots)$ of nl-elements is given by $K_1 = 1$ and $K_i = g(K_{i-1})$ for all $i > 1$.

**Corollary 1.** For $k > 1$, the number of natural numbers with $\mathbb{E}_q$-representations of length $k$ is given by $K_{k+1} - K_k$. There are $K_2 = p$ such representations of length 1 (this includes the natural number 0).

**Corollary 2.** Let $(K_1, K_2, \ldots)$ be the sequence of nl-elements. Then for $k \geq 2$, $K_{k+1} - K_k = p a_k$ where $a_1 = 1$ and

$$a_{n+1} = \left\lfloor \frac{p-q}{q} \cdot \sum_{i=1}^{n} a_i \right\rfloor.$$

3.1. Rational Fibonacci Representations

Fix $m > 1$. By definition, we have $F_{m+1} > F_m$, and it is well known that $\gcd(F_{m+1}, F_m) = 1$. Consequently, we can consider $\mathbb{E}_q$-representations where $p =
$F_{m+1}$ and $q = F_m$. For the remainder of this section, we let $p = F_{m+1}$ and $q = F_m$ and call the associated $\frac{p}{q}$-representations simply $F_m$-representations. The following lemma allows us to prove Theorem 2.

**Lemma 2.** Let $m, k \in \mathbb{N}$ with $k < m-2$. The $F_m$-representation of $F_{m+1}(F_{k+3}-1)$ is given by $(n_0, n_1, \ldots, n_{k+1})_{\frac{p}{q}}$ where $n_0 = 0$, and

$$n_i := F_{m-1} + (-1)^{k+1-i}F_{m-k+i-3}$$

for each $1 \leq i \leq k + 1$. Furthermore $K_{k+2} = F_{m+1}(F_{k+3}-1)$.

**Proof.** Let $n = (n_0, n_1, \ldots, n_{k+1})_{\frac{p}{q}}$. First, we note that $n_0 = 0$ and we check that $n_{k+1} = F_{m-1} + F_{m-2} = F_m$. Also, since $k < m - 2$ and $i \leq k + 1$ we have $0 \leq F_{m-k+i-3} \leq F_m$. Thus, we see that

$$0 \leq F_{m-1} - F_{m-k+i-3} \leq n_i \leq F_{m-1} + F_{m-k+i-3} < F_{m-1} + F_{m-2} = F_m$$

for $1 \leq i \leq k$. According to Lemma 1, $n = K_{k+2}$, and Theorem 2 implies that $n = F_{m+1}(F_{k+3}-1)$.

**Proof of Theorem 2.** Let $n \in \mathbb{N}$ with $n \geq 1$. Then, choose $M = n + 3$. Then for any $m \geq M$ consider the $F_m$-representations of natural numbers and the associated sequence of $n$-elements. According to Lemma 2, we have $K_{k+2} = F_{m+1}(F_{k+3}-1)$ for all $1 \leq k \leq n$. In particular, we have

$$K_{n+1} - K_n = F_{m+1}(F_{n+2} - 1) - F_{m+1}(F_{n+1} - 1) = F_{m+1}(F_{n+2} - F_{n+1}) = F_{m+1}F_n.$$ 

Furthermore, by Corollary 2, we have

$$K_{n+1} - K_n = F_{m+1}A_n^m$$

where $A^m$ is defined in equation (1). Since $F_{m+1} \neq 0$, we have $F_n = A_n^m$. By definition $A_1^m = F_1$ and so the result holds.

Therefore, the family of sequences $\{A^m\}$ converges pointwise to $F$. Moreover, by Corollary 1 and Corollary 2, we see that $A^m$ counts the number of multiples of $p = F_{m+1}$ having $F_m$-representations of length $k$, giving these sequences a combinatorial interpretation. Moreover, in terms of the tree $I_{p/q}$ defined in [1], the sequence $A^m$ gives the number of vertices at fixed distances from the root. Finally, it can be checked (using methods similar to those describing equation 2 in the introduction) that $A^m \neq F$ for all $m$.

**Acknowledgements.** We would like to thank the anonymous referee for helpful comments that clarified and shortened this note.
References


