Unipotent Brauer Character Values of $GL(n, \mathbb{F}_q)$ and the Forgotten Basis of the Hall Algebra

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Abstract. We give a formula for the values of irreducible unipotent $p$-modular Brauer characters of $GL(n, \mathbb{F}_q)$ at unipotent elements, where $p$ is a prime not dividing $q$, in terms of (unknown!) weight multiplicities of quantum $GL_n$ and certain generic polynomials $S_{\lambda, \mu}(q)$. These polynomials arise as entries of the transition matrix between the renormalized Hall-Littlewood symmetric functions and the forgotten symmetric functions. We also provide an alternative combinatorial algorithm working in the Hall algebra for computing $S_{\lambda, \mu}(q)$.

Keywords: symmetric function, general linear group, unipotent representation, Brauer character

1. Introduction

In the character theory of the finite general linear group $G_n = GL(n, \mathbb{F}_q)$, the Gelfand-Graev character $\Gamma_n$ plays a fundamental role. By definition [5], $\Gamma_n$ is the character obtained by inducing a “general position” linear character from a maximal unipotent subgroup. It has support in the set of unipotent elements of $G_n$ and for a unipotent element $u$ of type $\lambda$ (i.e. the block sizes of the Jordan normal form of $u$ are the parts of the partition $\lambda$) Kawanaka [7, 3.2.24] has shown that

$$\Gamma_n(u) = (-1)^h(1 - q)(1 - q^2)\cdots(1 - q^{h(\lambda)}),$$

where $h(\lambda)$ is the number of non-zero parts of $\lambda$. The starting point for this article is the problem of calculating the operator determined by Harish-Chandra multiplication by $\Gamma_n$.

We have restricted our attention throughout to character values at unipotent elements, when it is convenient to work in terms of the Hall algebra, that is [13, Section 10.1], the vector space $g = \bigoplus_{n \geq 0} g_n$, where $g_n$ denotes the set of unipotent-supported $\mathbb{C}$-valued class functions on $G_n$, with multiplication coming from the Harish-Chandra induction operator. For a partition $\lambda$ of $n$, let $\pi_\lambda \in g_n$ denote the class function which is 1 on unipotent elements of type $\lambda$ and zero on all other conjugacy classes of $G_n$. Then, $\{\pi_\lambda\}$ is a basis for the Hall algebra labelled by all partitions. Let $\gamma_n : g \to g$ be the linear operator determined by multiplication in $g$ by $\Gamma_n$. We describe in Section 2 an explicit recursive algorithm, involving the combinatorics of addable and removable nodes, for calculating the effect of $\gamma_n$ on the

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basis \( \{ \pi_\lambda \} \). As an illustration of the algorithm, we rederive Kawanaka’s formula (1.1) in 2.12.

Now recall from [13] that \( g \) is isomorphic to the algebra \( \Lambda_\mathbb{C} \) of symmetric functions over \( \mathbb{C} \), the isomorphism sending the basis element \( \pi_\lambda \) of \( g \) to the Hall-Littlewood symmetric function \( \tilde{P}_\lambda \in \Lambda_\mathbb{C} \) (renormalized as in [9, section II.3, ex. 2]). Consider instead the element \( \vartheta_\lambda \in g \) which maps under this isomorphism to the forgotten symmetric function \( f_\lambda \in \Lambda_\mathbb{C} \) (see [9, section I.2]). Introduce the renormalized Gelfand-Graev operator \( \gamma_\lambda = \delta \circ \vartheta_\lambda \), where \( \delta : g \to g \) is the linear map with \( \delta(\pi_\lambda) = \frac{1}{q^{n_\lambda}} \pi_\lambda \) for all partitions \( \lambda \). We show in Theorem 3.5 that

\[
\vartheta_\lambda = \sum_{(n_1, \ldots, n_h)} \gamma_{n_1} \circ \gamma_{n_2} \circ \cdots \circ \gamma_{n_h}(\pi_{(0)}),
\]

summing over all \((n_1, \ldots, n_h)\) obtained by reordering the non-zero parts \( \lambda_1, \ldots, \lambda_h \) of \( \lambda \) in all possible ways. Thus, we obtain a direct combinatorial construction of the ‘forgotten basis’ \( \{ \vartheta_\lambda \} \) of the Hall algebra.

Let \( K = (K_{\lambda, \mu}) \) denote the matrix of Kostka numbers [9, I, (6.4)], \( \tilde{K} = (\tilde{K}_{\lambda, \mu}(q)) \) denote the matrix of Kostka-Foulkes polynomials (renormalized as in [9, III, (7.11)]) and \( J = (J_{\lambda, \mu}) \) denote the matrix with \( J_{\lambda, \mu} = 0 \) unless \( \mu = \lambda' \) when it is 1, where \( \lambda' \) is the conjugate partition to \( \lambda \). Consulting [9, section I.6, section III.6], the transition matrix between the bases \( \{ \pi_\lambda \} \) and \( \{ \vartheta_\lambda \} \), i.e. the matrix \( S = (S_{\lambda, \mu}(q)) \) of coefficients such that

\[
\vartheta_\lambda = \sum_{\mu} S_{\lambda, \mu}(q) \pi_\mu,
\]

is then given by the formula \( S = K^{-1}J\tilde{K} \); in particular, this implies that \( S_{\lambda, \mu}(q) \) is a polynomial in \( q \) with integer coefficients. Our alternative approach to computing \( \vartheta_\lambda \), using (1.2) allows explicit computation of the polynomials \( S_{\lambda, \mu}(q) \) in some extra cases (e.g. when \( \mu = (1^n) \)) not easily deduced from the matrix product \( K^{-1}J\tilde{K} \).

To explain our interest in this, let \( \chi_\lambda \) denote the irreducible unipotent character of \( G_\mathbb{C} \) labelled by the partition \( \lambda \), as constructed originally in [12], and let \( \sigma_\lambda \in g \) denote its projection to unipotent-supported class functions. So, \( \sigma_\lambda \) is the element of \( g \) mapping to the Schur function \( s_\lambda \) under the isomorphism \( g \to \Lambda_\mathbb{C} \) (see [13]). Since \( \sigma_\lambda = \sum_\mu K_{\lambda, \mu} \vartheta_\mu \) [9, section I.6], we deduce that the value of \( \chi_\lambda \) at a unipotent element \( u \) of type \( \nu \) can be expressed in terms of the Kostka numbers \( K_{\lambda, \mu} \) and the polynomials \( S_{\mu, \nu}(q) \) as

\[
\chi_\lambda(u) = \sum_\mu K_{\lambda, \mu} S_{\mu, \nu}(q).
\]

This is a rather clumsy way of expressing the unipotent character values in the ordinary case, but this point of view turns out to be well-suited to describing the irreducible unipotent Brauer characters.

So now suppose that \( p \) is a prime not dividing \( q \), \( k \) is a field of characteristic \( p \) and let the multiplicative order of \( q \) modulo \( p \) be \( \ell \). In [6], James constructed for each partition \( \lambda \) of \( n \) an absolutely irreducible, unipotent \( kG_\ell \)-module \( D_\lambda \) (denoted \( L(1, \lambda) \) in [1]), and showed
that the set of all $D_s$ gives the complete set of non-isomorphic irreducible modules that arise as constituents of the permutation representation of $\mathbb{C}G_n$ on cosets of a Borel subgroup. Let $\chi^p_s$ denote the Brauer character of the module $D_s$, and $\sigma^p_s \in g$ denote the projection of $\chi^p_s$ to unipotent-supported class functions. Then, as a direct consequence of the results of Dipper and James [3], we show in Theorem 4.6 that 

$$\chi^p_s(\tau) = \sum_{\mu} K^{p,\ell}_{h,\mu} S_{\mu,\tau}(q)$$

This formula reduces the problem of calculating the values of the irreducible unipotent Brauer characters at unipotent elements to knowing the modular Kostka numbers $K^{p,\ell}_{h,\mu}$ and the polynomials $S_{\mu,\tau}(q)$.

Most importantly, taking $\tau = 1^n$ in (1.5), we obtain the degree formula:

$$\chi^p_s(1) = \sum_{\mu} K^{p,\ell}_{h,\mu} S_{\mu,1^n}(q)$$

where, as a consequence of (1.2) (see Example 3.7),

$$S_{\mu, 1^n}(q) = \sum_{(n_1, \ldots, n_h)} \left[ \prod_{i=1}^{h} (q^{n_i} - 1) \prod_{i=1}^{h} (q^{n_i + \cdots + n_h} - 1) \right]$$

summing over all $(n_1, \ldots, n_h)$ obtained by reordering the non-zero parts $\mu_1, \ldots, \mu_h$ of $\mu$ in all possible ways. This formula was first proved in [1, section 5.5], as a consequence of a result which can be regarded as the modular analogue of Zelevinsky’s branching rule [13, section 13.5] involving the affine general linear group. The proof presented here is independent of [1] (excepting some self-contained results from [1, section 5.1]), appealing instead directly to the original characteristic 0 branching rule of Zelevinsky, together with the work of Dipper and James on decomposition matrices. We remark that since all of the integers $S_{\mu, 1^n}(q)$ are positive, the formula (1.6) can be used to give quite powerful lower bounds for the degrees of the irreducible Brauer characters, by exploiting a $q$-analogue of the Premet-Suprunenko bound for the $K^{p,\ell}_{h,\mu}$. The details can be found in [2].

To conclude this introduction, we list in the table below the polynomials $S_{\mu,\tau}(q)$ for $n \leq 4$:
Finally, I would like to thank Alexander Kleshchev for several helpful discussions about this work.

2. An algorithm for computing $\gamma_n$

We will write $\lambda \vdash n$ to indicate that $\lambda$ is a partition of $n$, that is, a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of non-negative integers summing to $n$. Given $\lambda \vdash n$, we denote its Young diagram by $[\lambda]$; this is the set of nodes

$$\{ (i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq j \leq \lambda_i \}.$$ 

By an addable node (for $\lambda$), we mean a node $A \in \mathbb{N} \times \mathbb{N}$ such that $[\lambda] \cup \{ A \}$ is the diagram of a partition; we denote the new partition obtained by adding the node $A$ to $\lambda$ by $\lambda \cup A$. By a removable node (for $\lambda$) we mean a node $B \in [\lambda]$ such that $[\lambda] \setminus \{ B \}$ is the diagram of a partition; we denote the new partition obtained by removing $B$ from $\lambda$ by $\lambda \setminus B$. The depth $d(B)$ of the node $B = (i, j) \in \mathbb{N} \times \mathbb{N}$ is the row number $i$. If $B$ is removable for $\lambda$, it will also be convenient to define $e(B)$ (depending also on $\lambda$) to be the depth of the next removable node above $B$ in the partition $\lambda$, or 0 if no such node exists. For example consider the partition $\lambda = (4, 4, 2, 1)$, and let $A, B, C$ be the removable nodes in order of increasing depth:

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Then, $e(A) = 0$, $e(B) = d(A) = 2$, $e(C) = d(B) = 3$, $d(C) = 4$.

Now fix a prime power $q$ and let $G_n$ denote the finite general linear group $GL(n, \mathbb{F}_q)$ as in the introduction. Let $V_n = \mathbb{F}_q^n$ denote the natural $n$-dimensional left $G_n$-module, with standard basis $v_1, \ldots, v_n$. Let $H_n$ denote the affine general linear group $AGL(n, \mathbb{F}_q)$. This is the semidirect product $H_n = V_n \rtimes G_n$ of $G_n$ acting on the elementary Abelian group $V_n$.

We always work with the standard embedding $H_n \hookrightarrow G_{n+1}$ that identifies $H_n$ with the subgroup of $G_{n+1}$ consisting of all matrices of the form:

$$\begin{bmatrix}
\ast & \ast \\
0 & \ldots & 0 & 1
\end{bmatrix}.$$
Thus, we have a chain of subgroups $1 = H_0 \subset G_1 \subset H_1 \subset G_2 \subset H_2 \subset \ldots$ (by convention, we allow the notations $G_0$, $H_0$ and $V_0$, all of which denote groups with one element.)

For $\lambda \vdash n$, let $u_{\lambda} \in G_n \subset H_n$ denote the upper uni-triangular matrix consisting of Jordan blocks of sizes $\lambda_1, \lambda_2, \ldots$ down the diagonal. As is well-known, $\{u_{\lambda} \mid \lambda \vdash n\}$ is a set of representatives of the unipotent conjugacy classes in $G_n$. We wish to describe instead the unipotent classes in $H_n$. These were determined in [10, section 1], but the notation here will be somewhat different. For $\lambda \vdash n$ and an addable node $A$ for $\lambda$, define the upper uni-triangular $(n+1) \times (n+1)$ matrix $u_{\lambda,A} \in H_n \subset G_{n+1}$ by

$$u_{\lambda,A} = \begin{cases} u_{\lambda} & \text{if } A \text{ is the deepest addable node}, \\ u_{\lambda_1+\ldots+\lambda_d,A} u_{\lambda} & \text{otherwise}. \end{cases}$$

If instead $\lambda \vdash (n+1)$ and $B$ is removable for $\lambda$ (hence addable for $\lambda \setminus B$) define $u_{\lambda,B}$ to be a shorthand for $u_{\lambda \setminus B,B} \in H_n$. To aid translation between our notation and that of [10], we note that $u_{\lambda,B}$ is conjugate to the element denoted $c_{n+1}(1^{(k)}, \mu)$ there, where $k = \lambda,_{(B)}$ and $\mu$ is the partition obtained from $\lambda$ by removing the $d(B)$th row. Then:

**Lemma 2.1**

(i) The set

$$\{u_{\lambda,A} \mid \lambda \vdash n, A \text{ addable for } \lambda\} = \{u_{\lambda,B} \mid \lambda \vdash (n+1), B \text{ removable for } \lambda\}$$

is a set of representatives of the unipotent conjugacy classes of $H_n$.

(ii) For $\lambda \vdash (n+1)$ and a removable node $B$, $|C_{G_{n+1}}(u_{\lambda})|/|C_{H_n}(u_{\lambda,B})| = q^{d(B)} - q^{c(B)}$.

**Proof:** Part (i) is a special case of [10, 1.3(i)], where all conjugacy classes of the group $H_n$ are described. For (ii), combine the formula for $|C_{G_{n+1}}(u_{\lambda})|$ from [12, 2.2] with [10, 1.3(i)], or calculate directly. □

For any group $G$, we write $C(G)$ for the set of $\mathbb{C}$-valued class functions on $G$. Let $\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}_n \subset \bigoplus_{n \geq 0} C(G_n)$ denote the Hall algebra as in the introduction. We recall that $\mathfrak{g}$ is a graded Hopf algebra in the sense of [13], with multiplication $\circ : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ arising from Harish-Chandra induction and comultiplication $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ arising from Harish-Chandra restriction, see [13, section 10.1] for fuller details. Also defined in the introduction, $\mathfrak{g}$ has the natural ‘characteristic function’ basis $\{\pi_{\lambda}\}$ labelled by all partitions.

By analogy, we introduce an extended version of the Hall algebra corresponding to the affine general linear group. This is the vector space $\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}_n$ where $\mathfrak{g}_n$ is the subspace of $C(H_n)$ consisting of all class functions with support in the set of unipotent elements of $H_n$, with algebra structure to be explained below. To describe a basis for $\mathfrak{g}$, given $\lambda \vdash n$ and an addable node $A$, define $\pi_{\lambda,A} \in C(H_n)$ to be the class function which takes value 1 on $u_{\lambda,A}$ and is zero on all other conjugacy classes of $H_n$. Given $\lambda \vdash (n+1)$ and a removable $B$, set $\pi_{\lambda,B} = \pi_{\lambda \setminus B,B}$. Then, in view of Lemma 2.1(i), $\{\pi_{\lambda,A} \mid \lambda \text{ a partition, } A \text{ addable for } \lambda\} = \{\pi_{\lambda,B} \mid \lambda \text{ a partition, } B \text{ removable for } \lambda\}$ is a basis for $\mathfrak{g}$.
Now we introduce various operators as in [13, section 13.1] (but take notation instead from [1, section 5.1]). First, for \( n \geq 0 \), we have the inflation operator

\[ e_0^n : C(G_n) \to C(H_n) \]

defined by \((e_0^n \chi)(vg) = \chi(g)\) for \( \chi \in C(G_n) \), \( v \in V_n \), \( g \in G_n \). Next, fix a non-trivial additive character \( \chi : F_q \to \mathbb{C}^\times \) and let \( \chi_n : V_n \to \mathbb{C}^\times \) be the character defined by \( \chi_n(\sum_{i=1}^n c_i v_i) = \chi(c_n) \). The group \( G_n \) acts naturally on the characters \( C(V_n) \) and one easily checks that the subgroup \( H_{n-1} < G_n \) centralizes \( \chi_n \). In view of this, it makes sense to define for each \( n \geq 1 \) the operator

\[ e_1^n : C(H_{n-1}) \to C(H_n), \]

namely, the composite of inflation from \( H_{n-1} \) to \( V_n H_{n-1} \) with the action of \( V_n \) being via the character \( \chi_n \), followed by ordinary induction from \( V_n H_{n-1} \) to \( H_n \). Finally, for \( n \geq 1 \) and \( 1 \leq i \leq n \), we have the operator

\[ e_i^n : C(G_{n-i}) \to C(H_n) \]

defined inductively by \( e_i^n = e_0^n \circ e_{i-1}^{n-1} \). The significance of these operators is due to the following lemma [13, section 13.2]:

**Lemma 2.2** The operator \( e_0^n \oplus e_1^n \oplus \cdots \oplus e_n^n : C(G_n) \oplus C(G_{n-1}) \oplus \cdots \oplus C(G_0) \to C(H_n) \) is an isometry.

We also have the usual restriction and induction operators

\[ \text{res}^{G_{n-i}}_{H_{n-1}} : C(G_n) \to C(H_{n-1}) \quad \text{ind}^{G_i}_{H_{n-1}} : C(H_{n-1}) \to C(G_n). \]

One checks that all of the operators \( e_0^n, e_1^n, \text{res}^{G_n}_{H_{n-1}}, \) and \( \text{ind}^{G_i}_{H_{n-1}} \) send class functions with unipotent support to class functions with unipotent support. So, we can define the following operators between \( g \) and \( h \), by restricting the operators listed to unipotent-supported class functions:

\[ e_+ : h \to h, \quad e_+ = \bigoplus_{n \geq 1} e_0^n; \tag{2.3} \]
\[ e_1 : g \to h, \quad e_1 = \bigoplus_{n \geq 1} e_1^n = (e_+)' \circ e_0; \tag{2.4} \]
\[ \text{ind} : h \to g, \quad \text{ind} = \bigoplus_{n \geq 1} \text{ind}^{G_n}_{H_{n-1}}; \tag{2.5} \]
\[ \text{res} : g \to h, \quad \text{res} = \bigoplus_{n \geq 0} \text{res}^{G_n}_{H_{n-1}} \tag{2.6} \]

where for the last definition, \( \text{res}^{G_n}_{H_{n-1}} \) should be interpreted as the zero map.
Now we indicate briefly how to make $h$ into a graded Hopf algebra. In view of Lemma 2.2, there are unique linear maps $\delta : h \to h$ and $\Delta : h \to h \otimes h$ such that

\[(e_i \chi) \circ (e_j \tau) = e_{i+j} (\chi \circ \tau), \quad (2.7)\]
\[\Delta(e_k \psi) = \sum_{i+j=k} (e_i \otimes e_j) \Delta(\psi), \quad (2.8)\]

for all $i, j, k \geq 0$ and $\chi, \tau, \psi \in g$. One can check, using Lemma 2.2 and the fact that $g$ is a graded Hopf algebra, that these operations endow $h$ with the structure of a graded Hopf algebra (the unit element is $e_0$...0, and the counit is the map $e_i \chi \mapsto \delta_i \eta(e_i)$ where $\eta : g \to \mathbb{C}$ is the counit of $g$). Moreover, the map $e_0 : g \to h$ is a Hopf algebra embedding. Unlike for the operations of $g$, we do not know of a natural representation theoretic interpretation for these operations on $h$ except in special cases, see [1, section 5.2].

The effect of the operators (2.3)–(2.6) on our characteristic function bases is described explicitly by the following lemma:

Lemma 2.9 Let $\lambda \vdash n$ and label the addable nodes (resp. removable nodes) of $\lambda$ as $A_1, A_2, \ldots, A_s$ (resp. $B_1, B_2, \ldots, B_{s-1}$) in order of increasing depth. Also let $B = B_t$ be some fixed removable node. Then,

(i) $e_0 \pi_{\lambda} = \sum_{i=1}^{s} \pi_{\lambda, A_i}$;
(ii) $e_t \pi_{\lambda,B} = q^{\delta(B)} \sum_{i=t+1}^{s} \pi_{\lambda, A_i} - q^{\epsilon(B)} \sum_{i=t}^{s} \pi_{\lambda, A_i}$;
(iii) $\text{res} \pi_{\lambda,B} = \sum_{i=1}^{s-1} \pi_{\lambda, B_i}$;
(iv) $\text{ind} \pi_{\lambda, B} = (q^{\delta(B)} - q^{\epsilon(B)}) \pi_{\lambda}$;
(v) $\text{ind} \circ \text{res} \pi_{\lambda} = (q^{\delta(\lambda)} - 1) \pi_{\lambda}$.

Proof:

(i) For $\mu \vdash n$ and $A$ admissible, we have by definition that $(e_0 \pi_{\lambda})(u_{\mu,A}) = \pi_{\lambda}(u_{\mu}) = \delta_{\lambda,\mu}$. Hence, $e_0 \pi_{\lambda} = \sum_A \pi_{\lambda, A}$, summing over all admissible nodes $A$ for $\lambda$.
(ii) This is a special case of [10, 2.4] translated into our notation.
(iii) For $\mu \vdash n$ and $B$ removable, $(e_t \pi_{\lambda})(u_{\mu,B})$ is zero unless $u_{\mu,B}$ is conjugate in $G_n$ to $u_t$, when it is one. So the result follows on observing that $u_{\mu,B}$ is conjugate in $G_n$ to $u_t$ if and only if $\mu = \lambda$.
(iv) We can write $\pi_{\lambda, B} = \sum_{\mu \vdash n} c_{\mu,B} \pi_{\mu}$. To calculate the coefficient $c_{\mu,B}$ for fixed $\mu \vdash n$, we use (iii), Lemma 2.1(ii) and Frobenius reciprocity.
(v) This follows at once from (iii) and (iv) since $\sum_B (q^{\delta(B)} - q^{\epsilon(B)}) = q^{\delta(\lambda)} - 1$, summing over all removable nodes $B$ for $\lambda$.

Lemma 2.9(i, ii) give explicit formulae for computing the operator $e_n = (e_t)^n \circ e_0$. The connection between $e_n$ and the Gelfand-Graev operator $\gamma_0$ defined in the introduction comes from the following result:

Theorem 2.10 For $n \geq 1$, $\gamma_0 = \text{ind} \circ e_{n-1}$. 


Proof: In [1, Theorem 5.1e], we showed directly from the definitions that for any \( \chi \in C(G_m) \) and any \( n \geq 1 \), the class function \( \chi \Gamma_n \in C(G_{m+n}) \) obtained by Harish-Chandra induction from \( (\chi, \Gamma_n) \in C(G_m) \times C(G_n) \) is equal to \( \text{res}_{H_n}^{H_{m+n}}(e_{m+n}^\chi) \). Moreover, by [1, Lemma 5.1c(iii)], we have that \( \text{res}_{H_n}^{H_{m+n}} \circ e_{m+n}^\chi = \text{ind}_{H_n}^{H_{m+n}} \chi \). Hence,

\[
\chi \Gamma_n = \text{ind}_{H_n}^{H_{m+n}}(e_{m+n-1}^{\chi}).
\]

The theorem is just a restatement of this formula at the level of unipotent-supported class functions.

Example 2.11 We show how to calculate \( \gamma_2 \pi_{(3, 2)} \) using Lemma 2.9 and the theorem. We omit the label \( \pi \) in denoting basis elements, and in the case of the intermediate basis elements of \( h \), we mark removable nodes with \( \times \).

\[
\gamma_2 = \text{ind} \circ e_+ \left( \begin{array}{c}
x \\
q^2 \end{array} \right) + \begin{array}{c}
x \\
q^2 \end{array} + \begin{array}{c}
x \\
q^2 \end{array}
\]

\[
= \text{ind} \left( - \begin{array}{c}
x \\
q^2 \end{array} + (q-1) \begin{array}{c}
x \\
q^2 \end{array} + (q-1) \begin{array}{c}
x \\
q^2 \end{array} \right)
\]

\[
= -(q-1) \begin{array}{c}
x \\
q^2 \end{array} + (q-1)(q^2-q-1) \begin{array}{c}
x \\
q^2 \end{array} + (q-1)(q^2-q^2) \begin{array}{c}
x \\
q^2 \end{array} \]

\[
= -(q-1) \begin{array}{c}
x \\
q^2 \end{array} + (q-1)(q^2-q-1) \begin{array}{c}
x \\
q^2 \end{array} + (q-1)(q^2-q^2) \begin{array}{c}
x \\
q^2 \end{array}
\]

Example 2.12 We apply Theorem 2.10 to rederive the explicit formula (1.1) for the Gelfand-Graev character \( \Gamma_n \) itself. Of course, by Theorem 2.10, \( \Gamma_n = \text{ind} \circ \eta_{n-1}(\pi_{(0)}) \). We will in fact prove that

\[
e_{n-1}(\pi_{(0)}) = (-1)^{n-1} \sum_{\lambda \nsubseteq n} \sum_{B \text{ removable}} (1-q)(1-q^2) \cdots (1-q^{b_\lambda-1}) \pi_{\lambda,B}
\]

Then (1.1) follows easily on applying \( \text{ind} \) using Lemma 2.9(iv) and the calculation in the proof of Lemma 2.9(v).
To prove (2.13), use induction on \( n, n = 1 \) being immediate from Lemma 2.9(i). For \( n > 1 \), fix some \( \lambda \vdash n \), label the addable and removable nodes of \( \lambda \) as in Lemma 2.9 and take \( 1 \leq r \leq s \). Thanks to Lemma 2.9(ii), the \( \pi_{\lambda,\mathbf{A}_r} \)-coefficient of \( e_n(\pi_{(0)}) = e_{s} \circ e_{n-1}(\pi_{(0)}) \) only depends on the \( \pi_{\lambda,\mathbf{B}_i} \)-coefficients of \( e_{n-i}(\pi_{(0)}) \) for \( 1 \leq i \leq \min(r, s - 1) \). So by the induction hypothesis the \( \pi_{\lambda,\mathbf{A}_r} \)-coefficient of \( e_n(\pi_{(0)}) \) is the same as the \( \pi_{\lambda,\mathbf{A}_r} \)-coefficient of 

\[
(-1)^{n-1}(1 - q) \cdots (1 - q^{h(\lambda)-1}) \sum_{i=1}^{\min(r,s-1)} e_i \pi_{\lambda,\mathbf{B}_i},
\]

which using Lemma 2.9(ii) equals

\[
(-1)^{n-1}(1 - q) \cdots (1 - q^{h(\lambda)-1}) \sum_{i=1}^{\min(r,s-1)} \delta_{r,i} q^{\ell(\mathbf{B}_i)} - q^{\ell(\mathbf{B}_i)}.
\]

This simplifies to \((-1)^n(1 - q) \cdots (1 - q^{h(\lambda)-1})\) if \( r < s \) and \((-1)^n(1 - q) \cdots (1 - q^{h(\lambda)})\) if \( r = s \), as required to prove the induction step.

3. The forgotten basis

Recall from the introduction that for \( \lambda \vdash n \), \( \chi_{\lambda} \in \mathcal{C}(G_n) \) denotes the irreducible unipotent character parametrized by the partition \( \lambda \), and \( \sigma_{\lambda} \in \mathfrak{g} \) is its projection to unipotent-supported class functions. Also, \( \{\theta_{\lambda}\} \) denotes the ‘forgotten’ basis of \( \mathfrak{g} \), which can be defined as the unique basis of \( \mathfrak{g} \) such that for each \( n \) and each \( \lambda \vdash n \),

\[
\sigma_{\lambda} = \sum_{\mu \vdash n} K_{\lambda,\mu} \theta_{\mu}.
\] (3.1)

Given \( \lambda \vdash n \), we write \( \mu \vdash j \lambda \) if \( \mu \vdash (n - j) \) and \( \lambda_{i+1} \leq \mu_i \leq \lambda_i \) for all \( i = 1, 2, \ldots \). This definition arises in the following well-known inductive formula for the Kostka number \( K_{\lambda,\mu} \), i.e. the number of standard \( \lambda \)-tableaux of weight \( \mu \) [9, section I(6.4)]:

Lemma 3.2 For \( \lambda \vdash n \) and any composition \( \nu \vdash n \), \( K_{\lambda,\nu} = \sum_{\mu \vdash \nu} K_{\mu,\hat{\nu}} \), where \( j \) is the last non-zero part of \( v \) and \( \hat{\nu} \) is the composition obtained from \( \nu \) by replacing this last non-zero part by zero.

We will need the following special case of Zelevinsky’s branching rule [13, section 13.5] (see also [1, Corollary 5.4d(ii)] for its modular analogue):

Theorem 3.3 (Zelevinsky) For \( \lambda \vdash n \), \( \text{res}_{H_n}^{G_n} \chi_{\lambda} = \sum_{j \geq 1} \sum_{\mu \vdash j} e_{j-1} \chi_{\mu} \).

Now define the map \( \delta : \mathfrak{g} \to \mathfrak{g} \) as in the introduction by setting \( \delta(\pi_{\lambda}) = (q^{1/\min(\lambda_i)})\pi_{\lambda} \) for each partition \( \lambda \) and extending linearly to all of \( \mathfrak{g} \). The significance of \( \delta \) is that by Lemma 2.9(v),
\[ \hat{\gamma}_\lambda = \sum_{(n_1, \ldots, n_h)} \hat{\gamma}_{n_1} \circ \cdots \circ \hat{\gamma}_{n_h} \circ \hat{\gamma}_{n_1} \]  

(3.4)

summing over all compositions \((n_1, \ldots, n_h)\) obtained by reordering the \(h = h(\lambda)\) non-zero parts of \(\lambda\) in all possible ways.

**Theorem 3.5**  
*For any \(\lambda \vdash n\), \(\nabla_\lambda = \hat{\gamma}_\lambda(\pi(0))\).*

**Proof:**  
We will show by induction on \(n\) that
\[ \sigma_\lambda = \sum_{\mu \vdash n} K_{\lambda, \mu} \hat{\gamma}_\mu(\pi(0)). \]  

(3.6)

The theorem then follows immediately in view of the definition \((3.1)\) of \(\nabla_\lambda\). Our induction starts trivially with the case \(n = 0\). So now suppose that \(n > 0\) and that \((3.6)\) holds for all smaller \(n\). By Theorem 3.3,
\[ \text{res} \sigma_\lambda = \sum_{j \geq 1} \sum_{\mu \vdash j} e_{j-1} \sigma_{\mu^c}. \]

Applying the operator \(\delta \circ \text{ind}\) to both sides, we deduce that
\[ \sigma_\lambda = \sum_{j \geq 1} \sum_{\mu \vdash j} \hat{\gamma}_j(\sigma_{\mu^c}) = \sum_{j \geq 1} \sum_{\mu \vdash j} \sum_{v \vdash (n-j)} K_{\mu, v} \hat{\gamma}_j \circ \hat{\gamma}_v(\pi(0)). \]

(we have applied the induction hypothesis)
\[ = \sum_{j \geq 1} \sum_{\mu \vdash j} \sum_{v \vdash (n-j)} \sum_{(n_1, \ldots, n_h)} K_{\mu, (n_1, \ldots, n_h)} \hat{\gamma}_j \circ \hat{\gamma}_{n_1} \circ \cdots \circ \hat{\gamma}_{n_h} \circ \hat{\gamma}_{n_1} (\pi(0)) \]

(summing over \((n_1, \ldots, n_h)\) obtained by reordering the non-zero parts \(v\) in all possible ways)
\[ = \sum_{j \geq 1} \sum_{v \vdash (n-j)} \sum_{(n_1, \ldots, n_h)} K_{\mu, (n_1, \ldots, n_h, j)} \hat{\gamma}_j \circ \hat{\gamma}_{n_1} \circ \cdots \circ \hat{\gamma}_{n_h} \circ \hat{\gamma}_{n_1} (\pi(0)) \]

(we have applied Lemma 3.2)
\[ = \sum_{\eta \vdash n} \sum_{(m_1, \ldots, m_k)} K_{\lambda, (m_1, \ldots, m_k)} \hat{\gamma}_{m_1} \circ \cdots \circ \hat{\gamma}_{m_k} \circ \hat{\gamma}_{m_1} (\pi(0)) \]

(now summing over \((m_1, \ldots, m_k)\) obtained by reordering the non-zero parts of \(\eta\) in all possible ways)
\[ = \sum_{\eta \vdash n} K_{\lambda, \eta} \hat{\gamma}_\eta(\pi(0)) \]

which completes the proof. \(\square\)
Example 3.7 For \( \chi \in \mathfrak{g}_m \), write \( \deg \chi \) for its value at the identity element of \( G_n \). We wish to derive the formula (1.7) for \( \deg \theta_\lambda = S_{\lambda,(v)}(q) \) using Theorem 3.5. So, fix \( \lambda \vdash n \). Then, by Theorem 3.5,
\[
\deg \theta_\lambda = \sum_{(\nu_1, \ldots, \nu_k)} \deg \left[ \hat{\gamma}_n \circ \cdots \circ \hat{\gamma}_n(\pi_{(0)}) \right].
\] (3.8)
We will show that given \( \chi \in \mathfrak{g}_m \),
\[
\deg \hat{\gamma}_n(\chi) = (q^{m+n-1} - 1)(q^{m+n-2} - 1) \cdots (q^{m+1} - 1) \deg \chi;
\] (3.9) then the formula (1.7) follows easily from (3.8). Now \( \gamma_n \) is Harish-Chandra multiplication by \( \Gamma_n \), so
\[
\deg \gamma_n(\chi) = \deg \Gamma_n \deg \chi \cdot \frac{(q^{m+n} - 1) \cdots (q^{m+1} - 1)}{(q^n - 1) \cdots (q - 1)},
\] the last term being the index in \( G_{m+n} \) of the standard parabolic subgroup with Levi factor \( G_m \times G_n \). This simplifies using (1.1) to \( (q^{m+n} - 1) \cdots (q^{m+1} - 1) \deg \chi \). Finally, to calculate \( \deg \hat{\gamma}_n(\chi) \), we need to rescale using \( \delta \), which divides this expression by \( (q^{m+n} - 1) \).

4. Brauer character values

Finally, we derive the formula (1.5) for the unipotent Brauer character values. So let \( p \) be a prime not dividing \( q \) and \( \chi^p_{\lambda} \) be the irreducible unipotent \( p \)-modular Brauer character labelled by \( \lambda \) as in the introduction. Writing \( C^p(G_n) \) for the \( \mathbb{C} \)-valued class functions on \( G_n \) with support in the set of \( p \)-elements of \( G_n \), we view \( \chi^p_{\lambda} \) as an element of \( C^p(G_n) \). Let \( \hat{\chi}_\lambda \) denote the projection of the ordinary unipotent character \( \chi_\lambda \) to \( C^p(G_n) \). Then, by [6], we can write
\[
\hat{\chi}_\lambda = \sum_{\mu \vdash n} D_{\lambda,\mu} \chi^p_{\mu},
\] (4.1) and the resulting matrix \( D = (D_{\lambda,\mu}) \) is the unipotent part of the \( p \)-modular decomposition matrix of \( G_n \). One of the main achievements of the Dipper-James theory from [3] (see e.g. [1, (3.5a)]) relates these decomposition numbers to the decomposition numbers of quantum \( GL_n \).

To recall some definitions, let \( k \) be a field of characteristic \( p \) and \( v \in k \) be a square root of the image of \( q \) in \( k \). Let \( U_v \) denote the divided power version of the quantized enveloping algebra \( U_q(\mathfrak{gl}_n) \) specialized over \( k \) at the parameter \( v \), as defined originally by Lusztig [8] and Du [4, section 2] (who extended Lusztig’s construction from \( \mathfrak{sl}_n \) to \( \mathfrak{gl}_n \)). For each partition \( \lambda \vdash n \), there is an associated irreducible polynomial representation of \( U_v \) of high-weight \( \lambda \), which we denote by \( L(\lambda) \). Also let \( V(\lambda) \) denote the standard (or Weyl)
module of high-weight $\lambda$. Write

$$\text{ch } V(\lambda) = \sum_{\mu \vdash n} D'_{\lambda, \mu} \text{ch } L(\mu),$$  \hspace{1cm} (4.2)$$

so $D' = (D'_{\lambda, \mu})$ is the decomposition matrix for the polynomial representations of quantum $GL_n$ of degree $n$. Then, by [3]:

**Theorem 4.3 (Dipper and James)** \hspace{0.5cm} $D'_{\lambda, \mu} = D_{\lambda, \mu}$. \hspace{1cm} (4.3)

Let $\sigma^p_\mu$ denote the projection of $\chi^p_\mu$ to unipotent-supported class functions. The $\{\sigma^p_\mu\}$ also give a basis for the Hall algebra $g$. Inverting (4.1) and using (3.1),

$$\sigma^p_\mu = \sum_{\mu \vdash n} D^{-1}_{\lambda, \mu} \sigma^p_\mu = \sum_{\mu \vdash n} D^{-1}_{\lambda, \mu} K_{\mu, \nu} \theta_\nu,$$  \hspace{1cm} (4.4)$$

where $D^{-1} = (D^{-1}_{\lambda, \mu})$ is the inverse of the matrix $D$. On the other hand, writing $K^{p, \ell}_{\lambda, \mu}$ for the multiplicity of the $\mu$-weight space of $L(\lambda)$, and recalling that $K_{\lambda, \mu}$ is the multiplicity of the $\mu$-weight space of $V(\lambda)$, we have by (4.2) that

$$K_{\mu, \nu} = \sum_{p \vdash n} D'_{\mu, \nu} K^{p, \ell}_{\lambda, \mu}.$$  \hspace{1cm} (4.5)$$

Substituting (4.5) into (4.4) and applying Theorem 4.3, we deduce:

**Theorem 4.6** \hspace{0.5cm} $\sigma^p_\mu = \sum_{\mu \vdash n} K^{p, \ell}_{\lambda, \mu} \theta_\mu$. \hspace{1cm} (4.6)$$

Now (1.5) and (1.6) follow at once. This completes the proof of the formulae stated in the introduction.

**References**


