On Some Quasiconvex Functions with Linear Growth

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We establish (i) that the quasiconvexification of the distance function to any closed (possibly unbounded) subset of the space of conformal matrices $E_\partial$ in $M^{2\times 2}$ is bounded from below by the distance function itself, that is, $Q \text{ dist}(\cdot, K) \geq c \text{ dist}(\cdot, K)$, where $c > 0$ is a constant independent of $K$; (ii) some estimates of quasiconvexifications of the distance function to a closed subset of $M^{2\times 2}$ which is ‘supported’ by $E_\partial$; (iii) $Q \text{ dist}^p(\cdot, K) = Q \text{ dist}^p(\cdot, Q_p(K))$ for any $p \geq 1$ and any closed $K \subset M^{N \times n}$; (iv) for some nonconvex $K \subset M^{2\times 2}$, $Q \text{ dist}(\cdot, K)$ is homogeneous of degree one, conjugate invariant and convex, and $Q_1(K) = C(K)$.

1. Introduction

In this note we study some nonconvex, non-negative quasiconvex functions with linear growth at infinity obtained by using quasiconvex relaxations of the distance function to a closed set in $M^{2\times 2}$. The zero sets of these quasiconvex functions can be unbounded. We also give some conditions such that a homogeneous quasiconvex function of degree one in $M^{2\times 2}$ is convex in some two dimensional subspaces.

More precisely, we show that for every closed subset $K$ of $E_\partial$ ($E_\Sigma$, respectively)-the space of conformal (anti-conformal, respectively) matrices in $M^{2\times 2}$ - the quasiconvexification of the distance function $\text{ dist}(\cdot, K)$ is bounded below by itself, that is,

$$c \text{ dist}(P, K) \leq Q \text{ dist}(P, K), \quad (1.1)$$

and the constant $c > 0$ is independent of $K$. From the definition of quasiconvex relaxation (see Definition 1.1 below), we have

$$Q \text{ dist}(P, K) \leq \text{ dist}(P, K).$$

Therefore, $Q \text{ dist}(P, K)$ is not convex if $K$ is not convex. If $K \subset E_\partial$ ($E_\Sigma$, respectively) is closed and non-convex, we show that $\text{ dist}(\cdot, K)$ is not rank-1 convex in $M^{2\times 2}$, justifying the non-trivialness of (1.1). We also obtain an estimate of the lower bound for $Q \text{ dist}(\cdot, K)$ for any closed set $K \subset M^{2\times 2}$ which is supported (the precise definition of a supporting space will be given later) by $E_\partial$ ($E_\Sigma$, respectively). In the case where $E_\partial$ is the supporting space of $K$, we have that

$$c \text{ dist}(P, K) - C|P_{E_\partial}(P)| \leq Q \text{ dist}(P, K)$$
for $P \in K$, where $P_{E_3}$ is the orthogonal projection from $M^{2\times 2}$ to $E_3$.

Motivated from [7], we also study the behaviour of a nonnegative homogeneous quasiconvex function $f : M^{2\times 2} \mapsto \mathbb{R}$ of degree 1 under the conjugate invariant condition (see [23]) which is a less restrictive condition than that of [7]. We show in Theorem 2.7 below that $f$ must be convex in certain two dimensional subspaces of $M^{2\times 2}$ while $f$ is not necessarily convex (Remark 2.8). This result seems only valid in $M^{2\times 2}$ because we need a lemma in [7] (see Prop. 1.6 below) which holds only in two dimensional spaces.

We focus on subsets of $E_3$ and $E_5$ in $M^{2\times 2}$ because of the following two reasons.

(1) The weak type $(1, 1)$ estimates for the projection $P_{E_3}(D \phi)$ is classical and is readily available in [18]. Therefore we do not need too much harmonic analysis preparation. In fact, it is possible to establish a more general version of Theorem 2.2 for any subspace $E$ of $M^{N \times n}$ under the assumption that $E$ does not have rank-one matrices [13]. However we need to establish a more general weak type $(1, 1)$ estimate for a special class of singular integral operators.

(2) In [23, 25], the connected subsets of $M^{2\times 2}$ were characterized and used to construct nonconvex, nonnegative quasiconvex with $p$-growth at infinity. It was proved in

[25] that in $M^{2\times 2}$, a closed connected set $K$ does not have rank-one connections if and only if $K$ is a Lipschitz graph of a mapping $f$ from a closed set of $E_3$ to $E_5$ (or from a closed set of $E_5$ to $E_3$ respectively), such that

$$[f(A) - f(B)] < |A - B|, \quad A \neq B.$$ 

It was established in [23] that for any $p \in (1, \infty)$, there exists some $c(p) > 0$, if $K$ is such a graph satisfying $|f(A) - f(B)| \leq k|A - B|$ and $k^p < c(p)$, then the quasiconvex relaxation $Q \text{ dist}^p(\cdot, K)$ satisfies

$$\{ P \in M^{2\times 2}, Q \text{ dist}^p(P, K) = 0 \} = K.$$ 

It turns out that $c(p) \to 0$ as $p \to 1$. This motivated the study of the limiting case, that is, the graphs are reduced to closed subsets in $E_3$ and $E_5$ respectively.

The existence of nonconvex, nonnegative quasiconvex functions with subquadratic and linear growth were established in [19] and [22] respectively, where the zero sets of the functions are compact. A result of Müller [17] shows that there exists a nontrivial homogeneous quasiconvex function of degree one. Yan [21] proved that the $p$-quasiconvex hull of the set $\mathbb{R}^+SO(n)$ is larger than itself for $p < n/2$ and $n > 2$ (the $p$-quasiconvex hull of $K \subset M^{N \times n}$ can be defined by $Q_p K = (Q f)^{-1}(0)$, where $Q f$ is the quasiconvexification (see Definition 1.1 below) of the function $f$, where $f(P) = \text{ dist}^p(P, K)$, $P \in M^{N \times n}$). This indicates that the quasiconvex relaxations of the distance function to an unbounded nonconvex set might be convex. It is known that the $n$-th quasiconvex hull of $\mathbb{R}^+SO(n)$ remains itself. Recently, Dacorogna [7] showed that if $f : M^{2\times 2} \mapsto \mathbb{R}$ is rank-one convex, positively homogeneous of degree one and in addition, $f$ is $SO(2)$ rotationally invariant in the sense that $f(RAS) = f(A)$ for $R, S \in SO(2)$, $A \in M^{2\times 2}$, then $f$ is necessarily convex. Under the less restricted condition that $f$ is conjugating invariant, that is $f(RAR^T) = f(A)$ for $R \in SO(2)$ and $A \in M^{2\times 2}$, in [23] it was established the existence of $p$-homogeneous, conjugating invariant, quasiconvex functions for any $p > 1$ with their zero sets of the form

$$C_P = \{ x E_1 + y E_2 + r P, (x, y) \in \mathbb{R}^2 \},$$
where $E_1, E_2$ is a basis of $E_3$, $r = \sqrt{x^2 + y^2}$ and $P \in E_3$ is a fixed matrix. More precisely, for any $p > 1$ there exists some $c(p) > 0$ ($\lim_{p \to 1^+} c(p) = 0$), whenever $|P| < c(p)$ then $Q \, \text{dist}^p(\cdot, C_P)$ is $p$-homogeneous, conjugate invariant with $C_P$ as its zero set.

These results imply that for an unbounded set $K \subset M^{2 \times 2}$ the existence of a non-negative quasiconvex function

$$f : M^{2 \times 2} \to \mathbb{R}_+,$$ $f^{-1}(0) = K,$ and $0 \leq f(P) \leq C|P|^p + C_1$

does depend on the behaviour of the set $K$ near infinity.

We will show in this note that for any $C_P$, the 1-quasiconvex hull of $C_P$ equals its convex hull, that is, $Q_1(C_P) = C(C_P)$ and $Q \, \text{dist}(\cdot, C_P) = \text{dist}(\cdot, C(C_P))$, hence $Q \, \text{dist}(\cdot, C_P)$ is convex. As a tool, though it stands on its own right, we establish the following identity for any $p \geq 1$ and any closed set $K \subset M^{N \times n}$:

$$Q \, \text{dist}^p(\cdot, K) = Q \, \text{dist}^p(\cdot, Q_P(K)).$$

Some results on lower semicontinuity of quasiconvex functionals in $BV$ spaces have been established recently [1, 11, 12]. The integrands used in that approach are quasiconvex functions with linear growth at infinity. As far as I know, very few examples are known for such functions besides those with compact zero sets [19, 22]. [17] provided the first example of nonconvex quasiconvex functions of linear growth with unbounded zero sets.

Quasiconvex relaxation of certain distance functions to a given set in the space of matrices is an important subject in the study of martensitic phase transitions and optimal design problems (see [5, 3, 4, 10, 14, 15]). As far as I know, explicit relaxation formulas are hard to obtain and there are only a few known examples [6, 8, 14, 15]. Hence an estimate of the lower bound of the quasiconvex relaxation will provide us useful information on the set itself and on the relaxed function. A result in the same spirit as those in this note was established in [24] for $SO(n)$, that is

$$c(n) \, \text{dist}^2(\cdot, SO(n)) \leq Q \, \text{dist}^2(P, SO(n)).$$

In order to state and prove our main results, we need some preparation.

We denote by $M^{N \times n}$ the space of all real $N \times n$ matrices, with $\mathbb{R}^{Nn}$ norm, $\text{meas}(U)$ is the Lebesgue measure of a measurable subset $U \subset \mathbb{R}^n$ and let

$$\text{dist}(Q, K) = \inf_{P \in K} |Q - P|$$

be the distance function from a point $Q \in M^{N \times n}$ to a set $K \subset M^{N \times n}$. From now on let $\Omega$ be a nonempty, open and bounded subset of $\mathbb{R}^n$. We denote by $Du$ the gradient of a (vector-valued) function $u$ and we define the space $C^1_0(\Omega, \mathbb{R}^N)$ in the usual way. If $K \subset M^{N \times n}$, let $C(K)$ be its convex hull. Define the spaces of conformal matrices $E_3$ and anti-conformal matrices $E_5$ as

$$E_3 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \ a, b \in \mathbb{R} \right\}, \quad E_5 = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \ a, b \in \mathbb{R} \right\}.$$

Let $f : M^{N \times n} \to \mathbb{R}$ be a continuous function. The following are some conditions related to weak lower semicontinuity of the integral $\int_{\Omega} f(Du(x)) \, dx$ (c.f. [2, 16, 6]).
(1) $f$ is rank-one convex if for each matrix $A \in M^{N \times n}$ and each rank-one matrix $B = a \otimes b \in M^{N \times n}$, the function $t \rightarrow f(A + tB)$ is convex.

(2) $f$ is quasiconvex at $A \in M^{N \times n}$ on $\Omega$, if for any smooth function $\phi : \Omega \rightarrow \mathbb{R}^N$ compactly supported in $\Omega$,

$$\int_\Omega f(A + D\phi(x)) dx \geq \int_\Omega f(A) dx$$

holds. $f$ is quasiconvex if it is quasiconvex at every $A \in M^{N \times n}$. The class of quasiconvex functions is independent of the choice of $\Omega$.

It is well-known that quasiconvexity implies rank-one convexity (cf. [2, 16, 6]) while rank-one convexity does not, in general, imply quasiconvexity [20].

To construct quasiconvex functions, we need the following

**Definition 1.1 ([6])**. Suppose that $f : M^{N \times n} \rightarrow \mathbb{R}$ is a continuous function. The quasiconvexification of $f$ is defined by

$$\sup\{g \leq f; g \text{ quasiconvex} \}$$

and will be denoted by $Qf$.

**Proposition 1.2 ([6])**. Suppose that $f : M^{N \times n} \rightarrow R$ is continuous, then

$$Qf(P) = \inf_{\phi \in C_0^0(\Omega; \mathbb{R}^N)} \frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} f(P + D\phi(x)) dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. In particular the infimum in (1.2) is independent of the choice of $\Omega$.

**Definition 1.3**. For a closed subset $K \subset M^{N \times n}$, we define the $p$-quasiconvex hull $Q_p(K)$ $(1 \leq p < \infty)$ as follows:

$$Q_p(K) = \{P \in M^{N \times n}, Q \operatorname{dist}^p(P, K) = 0\},$$

where $Q \operatorname{dist}^p(\cdot, K)$ is the quasiconvexification of $\operatorname{dist}^p(\cdot, K)$. If $K$ is compact, $Q_p(K)$ is independent of $p \geq 1$ [22]. However, this claim is not necessarily true if $K$ is unbounded (see [21]).

The following result is a special case of a more general theorem (see [9, pages 234, 236]).

**Proposition 1.4 (The measurable selection theorem)**. Let $B$ be a compact subset of $\mathbb{R}^p$ and $g$ a continuous function of $\mathbb{R}^n \times B$. Then, there exists a Lebesgue measurable mapping $\tilde{u} : \Omega \rightarrow B$ such that for all $x \in \Omega$:

$$g(x, \tilde{u}(x)) = \min_{a \in B} \{g(x, a)\}.$$
We need the following result established in [7],

**Proposition 1.6.** Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be such that

1. \( g(tx, ty) = tg(x, y) \) for every \( t \geq 0 \) and \( x, y \in \mathbb{R} \);
2. \( g \) is separately convex (i.e., \( g(x, \cdot) \) and \( g(\cdot, y) \) are convex for fixed \( x \) and fixed \( y \), respectively).

Then, \( g \) is convex in \( \mathbb{R}^2 \).

We conclude our preliminaries by giving a technical condition:

**Definition 1.7.** A non-empty, closed subset \( K \) of \( M^{2\times 2} \) is supported by \( E_\partial \) (\( E_3 \), respectively), if there exists an orthonormal basis of \( E_\partial \) (\( E_3 \), respectively) \( \{e_1, e_2\} \), such that \( e_i \cdot P \geq 0 \) for all \( P \in K \) and \( i = 1, 2 \), \( \cdot \) being the inner product of \( 2 \times 2 \) matrices.

We call \( E_\partial \) (\( E_3 \), respectively) the supporting space of \( K \).

2. **Statement of results**

**Lemma 2.1.** Suppose that \( K \subset E_\partial \) (\( E_3 \), respectively) is closed and non-convex. Then \( \text{dist}(\cdot, K) \) is not rank-1 convex.

**Theorem 2.2.** Suppose that \( K \subset E_\partial \) (\( E_3 \), respectively) is closed (possibly unbounded). Then, there exists a constant \( c > 0 \) independent of \( K \), such that

\[
\frac{c}{e} \text{dist}(P, K) \leq Q \text{dist}(P, K) \leq \text{dist}(P, K),
\]

for every \( P \in M^{2\times 2} \).

If we denote by

\[
K_\epsilon = \{ P \in M^{2\times 2}, \text{dist}(P, K) \leq \epsilon \},
\]

the \( \epsilon \)-neighbourhood of \( K \), we have the following simple consequence of Theorem 2.2.

**Corollary 2.3.** Under the assumption of Theorem 2.2,

\[
Q_1(K_\epsilon) \subset K_{c/\epsilon},
\]

for every \( \epsilon > 0 \), where \( c > 0 \) is the constant given by Theorem 2.2.

**Theorem 2.4.** Suppose that \( K \subset M^{2\times 2} \) is closed and is supported by \( E_\partial \) (\( E_3 \), respectively). Then, there exists a constant \( c > 0 \) independent of \( K \), such that

\[
\frac{c}{e} \text{dist}(P, K) - |P_{E_3}(P)| \leq Q \text{dist}(P, K),
\]

\[
(c \text{dist}(P, K) - |P_{E_\partial}(P)| \leq Q \text{dist}(P, K), \text{ respectively}).
\]

In particular,

\[
\frac{c}{e} \text{dist}(P, K) \leq Q \text{dist}(P, K)
\]

whenever \( P \in E_\partial \) (\( P \in E_3 \) respectively), which implies \( Q_1K \cap E_\partial = K \cap E_\partial \) ((\( Q_1K \cap E_3 = K \cap E_3 \), respectively).
Remark 2.5. A special case of [23, Th. 4.1] is that for every closed subset \( K \subseteq E_\partial \) \((E_\partial\) respectively), \( Q_p(K) = K \) for all \( p > 1 \). Combining Theorem 2.2 and that result, we see that \( Q_p(K) = K \) for all \( p \geq 1 \). The second statement in Theorem 2.4 implies that the inequality given by Theorem 2.2 holds on the supporting spaces, and the intersection of the \( 1 \)-quasiconvex hull with the supporting space does not enlarge the original intersection.

The following is a general result relating the \( p \)-quasiconvex hull of a closed set in \( M^{N \times n} \) and the quasiconvexification of the distance function. It might be a useful tool in the study of quasiconvexification of distance functions. We need this result here for the proof of Theorem 2.7 below.

Theorem 2.6. Let \( K \subseteq M^{N \times n} \) be non-empty and closed. Then

\[
Q \dist^p(\cdot, K) = Q \dist^p(\cdot, Q_p(K)),
\]

for every \( 1 \leq p < \infty \).

Theorem 2.7. Let \( f : M^{2 \times 2} \to \mathbb{R} \) be a nonnegative, \( 1 \)-homogeneous, conjugate invariant rank-one convex function. Let \( A \in E_\partial, B \in E_\partial \) be any fixed matrices and \( S(A, B) = \text{span}[A, B] \) be the subspace in \( M^{2 \times 2} \) spanned by \( A, B \). Then the restriction of \( f \) on \( S(A, B) \), \( f|_{S(A,B)} \) is convex.

Remark 2.8. In [17], the existence of a nonnegative homogeneous quasiconvex function of degree 1 was constructed which vanishes on the union of two one dimensional subspaces of \( E_\partial \). From Theorem 2.2, and the fact that every conformal matrix is conjugating invariant in the sense that \( RAR^T = A \), for \( R \in SO(2), A \in E_\partial \), we see that for every \( K \subseteq E_\partial, K \) is conjugating invariant. Therefore functions satisfying the assumptions of Theorem 2.7 are not necessarily convex on \( E_\partial \). This result is nearly optimal for the convexity of functions covered by Theorem 2.7.

Corollary 2.9. Let \( A, B \) and \( \text{span}[A, B] \) be as in Theorem 2.7. Suppose that \( K \subseteq \text{span}[A, B] \) is scaling invariant, that is, \( P \in K \) implies \( tP \in K \) for all \( t \geq 0 \). Then \( Q_1(K) = C(K) \) and \( Q \dist(\cdot, K) \) is convex.

Corollary 2.10. Let \( C_P \) be the cone based on \( E_\partial \):

\[
C_P = \{ xE_1 + yE_2 + rP, (x, y) \in \mathbb{R}^2 \},
\]

where \( E_1, E_2 \) is the basis of \( E_\partial \) defined by

\[
E_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\( r = \sqrt{x^2 + y^2} \) and \( P \in E_\partial \) is a fixed matrix. Then \( Q_1(C_P) = C(C_P) \) and \( Q \dist(\cdot, C_P) \) is homogeneous of degree one, conjugating invariant and convex.

Remark 2.11. If we assume that \(|P| < 1\) is sufficiently small, we have, (see [23]) that \( Q_2(C_P) = C_P \), hence \( Q_2(C_P) \neq Q_1(C_P) \). Corollary 2.10 provides another class of closed sets other than that given by Yan [21] such that the \( p \)-quasiconvex hull for an unbounded set may depend on \( p \).
3. Proofs of results

Proof of Lemma 2.1. Since $K \subset E_0$ is closed and not convex, we may find $A, B \in K$, such that the line segment $L = \{P = tA + (1-t)B, 0 < t < 1\}$ does not intersect $K$. Since $A$ and $B$ are conformal matrices, there exist $Q \in SO(2)$ and $a > 0$, such that $B - A = aQ$. Let

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we see that $aQJ \in E_7$ and $aQJ + aQ$ is a rank-1 matrix. If $\text{dist}(\cdot, K)$ was rank-1 convex, we would have

$$\text{dist}(A + \frac{1}{2}[aQ + aQJ], K) \leq \frac{1}{2}\text{dist}(A, K) + \frac{1}{2}\text{dist}(A + [aQ + aQJ], K).$$

Notice that $\text{dist}(A, K) = 0$. Since $A + \frac{1}{2}aQ \notin K$, we have

$$\text{dist}(A + \frac{1}{2}[aQ + aQJ], K) > |P_{E_5}(A + \frac{1}{2}[aQ + aQJ])| = \frac{\sqrt{2}}{2}a,$$

while because $P_{E_5}(A + [aQ + aQJ]) = B$, we have

$$\text{dist}(A + [aQ + aQJ], K) = |P_{E_5}(A + [aQ + aQJ])| = \sqrt{2}a.$$

Combining the above three inequalities, we see that $\frac{\sqrt{2}}{2}a < \sqrt{2}a$. This contradiction implies that $\text{dist}(\cdot, K)$ is not rank-one convex.

Proof of Theorem 2.2. We prove the result for $K \subset E_0$ only. The case for $E_5$ is similar. Notice that the upper bound in (2.1) is trivial because $Qf \leq f$ is always true (see (1.2)).

We use the weak type $(1,1)$ estimate for singular integral operators [18] as in [17]. For a fixed $P \in M^{2\times 2}$, we have, from Proposition 1.2, that there exists a sequence $(\phi_j)$ in $C_0^\infty(D, \mathbb{R}^2)$ such that

$$\lim_{j \to \infty} \int_D \text{dist}(P + D\phi_j(x), K)dx = Q\text{dist}(P, K) := a \geq 0,$$

(3.1)

where $D \subset \mathbb{R}^2$ is the unit square.

Let $P_{E_5}$ be the orthogonal projection from $M^{2\times 2}$ to $E_5$. Notice that $E_5$ is the orthogonal complement of $E_0$. Now, since $K \subset E_0$, we have

$$|P_{E_5}(A)| \leq \text{dist}(A, K)$$

for every $A \in M^{2\times 2}$.

If $Q \text{dist}(P, K) = a > 0$, we have, up to a subsequence,

$$\lim_{j \to \infty} \int_D |P_{E_5}(P + D\phi_j(x))|dx \leq a.$$
Since \(|P_{E\phi}(Q)|\) is a convex function in \(Q\), we have \(|P_{E\phi}(P)| \leq a\) so that
\[
\int_D |P_{E\phi}(D\phi_j(x))|dx \leq 2a + \delta_j,
\]
where \(\delta_j > 0\) and \(\delta_j \to 0\) as \(j \to \infty\). Notice that (3.2) implies
\[
\int_D \left( \left| \frac{\partial \phi_j^{(1)}(x)}{\partial x_1} - \frac{\partial \phi_j^{(2)}(x)}{\partial x_2} \right| + \left| \frac{\partial \phi_j^{(1)}(x)}{\partial x_2} + \frac{\partial \phi_j^{(2)}(x)}{\partial x_1} \right| \right) \leq 2a + \delta_j.
\]
Extending \(\phi_j\) outside \(D\) by zero and setting \(\psi_j = (\phi_j^{(1)}, -\phi_j^{(2)})\), we have
\[
\int_D [||\text{div} \psi_j|| + ||\text{curl} \psi_j||]dx \leq 2a + \delta_j.
\]
From the weak (1,1) type estimates in the singularity operator theory (see [18, Ch.2 and pp. 60]), we have
\[
\text{meas} \left\{ x \in \mathbb{R}^2, \ |D\psi_j(x)| > \lambda \right\} \leq \frac{C}{\lambda} \int_D [||\text{div} \psi_j|| + ||\text{curl} \psi_j||]dx \leq \frac{(2a + \delta_j)C}{\lambda},
\]
for every \(\lambda > 0\), where \(C > 0\) is a constant depending only on the operators \(\text{div}\) and \(\text{curl}\). Therefore, we have
\[
\text{meas} \left\{ x \in \mathbb{R}^2, \ |D\phi_j(x)| > \lambda \right\} \leq \frac{(2a + \delta_j)C}{\lambda}.
\]
Since the distance function \(\text{dist}(\cdot, K)\) satisfies
\[
|\text{dist}(A, K) - \text{dist}(B, K)| \leq |A - B|
\]
for \(A, B \in M^{2 \times 2}\), we see that
\[
\text{dist}(P, K) > \text{dist}(P + D\phi_j(x), K) + \lambda \quad \text{implies} \quad |D\phi_j(x)| > \lambda.
\]
In other words,
\[
D_\lambda := \{ x \in \Omega, \ \text{dist}(P, K) > \text{dist}(P + D\phi_j(x), K) + \lambda \} \subset \{ x \in \Omega, \ |D\phi_j(x)| > \lambda \},
\]
so that \(\text{meas}(D_\lambda) \leq \frac{(2a + \delta_j)C}{\lambda}\). Choosing \(\lambda = 2Ca + \sqrt{2Ca}\), we see that \((2a + \delta_j)C/\lambda < 1\) for sufficiently large \(j\). Hence,
\[
\int_D \text{dist}(P + D\phi_j(x), K)dx \geq \int_{D \setminus D_\lambda} \text{dist}(P + D\phi_j(x), K)dx
\geq |\text{dist}(P, K) - \lambda| \left( 1 - \frac{(2a + \delta_j)C}{\lambda} \right),
\]
for sufficiently large \(j > 0\). Passing to the limit in the above inequality, and noticing that \(\lim_{j \to \infty} \int_D \text{dist}(P + D\phi_j(x))dx = a\), we obtain
\[
a \geq |\text{dist}(P, K) - \lambda| \left( 1 - \frac{2aC}{\lambda} \right),
\]
(3.3)
which implies \( \text{dist}(P, K) \leq (2C + 1 + \sqrt{2C})a \). Letting \( c = (2C + 1 + \sqrt{2C})^{-1} \), we conclude that
\[
c \text{dist}(P, K) \leq a = Q \text{dist}(P, K).
\]
If \( a = 0 \), we let \( \lambda > 0 \) be any fixed number. For sufficiently large \( j > 0 \), we have \( C\delta_j < \lambda \). We then proceed as in the first case to obtain (3.3) with \( a = 0 \). Hence
\[
\text{dist}(P, K) \leq \lambda,
\]
for every \( \lambda > 0 \), thus \( \text{dist}(P, K) = 0 \). The proof is complete. \( \square \)

**Proof of Corollary 2.3.** Let \( P \in Q_1(K_\epsilon) \). Then \( Q \text{dist}(P, K_\epsilon) = 0 \), and since
\[
\text{dist}(P, K) \leq \text{dist}(P, K_\epsilon) + \epsilon,
\]
from Theorem 2.2 and inequality (3.4), we obtain
\[
c \text{dist}(P, K) \leq Q \text{dist}(P, K) \leq Q \text{dist}(P, K_\epsilon) + \epsilon = \epsilon
\]
which implies \( P \in K_{\epsilon/\epsilon} \). \( \square \)

**Proof of Theorem 2.4.** Similar to the proof of Theorem 2.2, we prove the theorem only in the case where \( K \) is supported by \( E_0 \). The proof for the other case is similar. Let \( P \in M^{2\times 2} \) be fixed and let \( \{\phi_j\} \) be a sequence in \( C_0^\infty(D, \mathbb{R}^2) \) such that
\[
\lim_{j \to \infty} \int_D \text{dist}(P + D\phi_j, K)dx = Q \text{dist}(P, K) = a \geq 0.
\]
For each fixed \( j > 0 \), since \( \phi_j \in C_0^\infty(D, \mathbb{R}^2) \), we have for some large \( R_j > 0 \),
\[
\text{dist}(P + D\phi_j(x), K) = \text{dist}(P + D\phi_j(x), K \cap \overline{B(0, R_j)}),
\]
\( \overline{B(0, R_j)} \) being the closed ball in \( M^{2\times 2} \), centred at the origin with radius \( R_j \). Now we apply Proposition 1.5 to the function \( F(x, Q) = |P + D\phi_j(x) - Q| \) for \( x \in \bar{D} \) and \( Q \in K \cap \overline{B(0, R_j)} \). There exists a measurable mapping \( X_j : \Omega \to K \cap \overline{B(0, R_j)} \), such that
\[
|P + D\phi_j(x) - X_j(x)| = \text{dist}(P + D\phi_j(x), K \cap \overline{B(0, R_j)}) = \text{dist}(P + D\phi_j(x), K),
\]
almost everywhere in \( \Omega \). Setting
\[
Y_j(x) = P_{E_0}(X_j(x)),
\]
we see from the assumption that \( E_0 \) is the supporting space of \( K \) that the components of \( Y_j \) do not change signs in \( \Omega \). Let
\[
\int_D \text{dist}(P + D\phi_j, K)dx = a + \delta_j,
\]
where $\delta_j \geq 0$ and $\lim_{j \to \infty} \delta_j = 0$. Since $\phi_j$ is zero on the boundary of $D$, we have that
\[
\begin{align*}
 a + \delta_j & = \int_D |P + D\phi_j(x) - X_j(x)|dx \\
 & \geq \int_D |P E_\delta(P + D\phi_j(x)) - Y_j(x)|dx \\
 & \geq \int_D \left( \sum_{i=1}^2 |e_i \cdot (P + D\phi_j(x) - Y_j(x))|^2 \right)^{1/2} dx \\
 & \geq \frac{1}{\sqrt{2}} \sum_{i=1}^2 \left| \int_D e_i \cdot (P - Y_j(x)) dx \right| \\
 & \geq \frac{1}{\sqrt{2}} \left( \sum_{i=1}^2 \left| \int_D e_i \cdot Y_j(x) dx \right| - |P E_\delta(P)| \right) \\
 & = \frac{1}{\sqrt{2}} \left( \sum_{i=1}^2 \left| \int_D e_i \cdot Y_j(x) dx \right| - |P E_\delta(P)| \right) \\
 & \geq \frac{1}{2} \int_D |Y_j(x)| dx - |P E_\delta(P)|.
\end{align*}
\]

The last inequality holds because the components of $Y_j(x)$ do not change signs. Hence we have
\[
\int_D |Y_j(x)| dx \leq 2(a + \delta_j + |P E_\delta(P)|).
\]

We also have
\[
\begin{align*}
 a + \delta_j & \geq \int_D |P E_\delta(P + D\phi_j(x)) - Y_j(x)| dx \\
 & \geq \int_D |P E_\delta(D\phi_j(x))| dx - \int_D |Y_j(x)| dx - |P E_\delta(P)|.
\end{align*}
\]

Combining this inequality and the previous one, we see that
\[
\int_D |P E_\delta(D\phi_j(x))| dx \leq 3(a + \delta_j + |P E_\delta(P)|).
\]

Similar to the argument as in the proof of Theorem 2.2, we have
\[
\text{meas}\left( \{ x \in D, |D\phi_j(x)| > \lambda \} \right) \leq \frac{C}{\lambda} \int_D |P E_\delta(D\phi_j(x))| dx \leq \frac{3C}{\lambda} (a + \delta_j + |P E_\delta(P)|),
\]
for every $\lambda > 0$. If $a > 0$ or $|P E_\delta(P)| > 0$, we choose
\[
\lambda = 6C(a + |P E_\delta(P)|)
\]
and, applying the same method as for Theorem 2.2, we have
\[
a + \delta_j \geq (\text{dist}(P, K) - \lambda) \left( 1 - \frac{6C}{\lambda} (a + \delta_j + |P E_\delta(P)|) \right).
\]
Passing to the limit $j \to \infty$ we obtain
\[
\text{dist}(P, K) \leq 6(C + 1)(Q \text{dist}(P, K) + |P_{E_{q}}(P)|).
\]

The proof is finished if we take $c = [6(C + 1)]^{-1}$.

If $a = 0$ and $|P_{E_{q}}(P)| = 0$, we see that $\int_{D} |Y_{j}| \, dx \to 0$. We may choose any fixed number $\lambda > 0$ and, following the proof for the case $a > 0$, we deduce that $\text{dist}(P, K) \leq \lambda$. The conclusion follows by letting $\lambda \to 0$.

Notice that if $e_{i} \cdot P \leq 0$, $i = 1, 2$, we may drop the term $|P_{E_{q}}(P)|$ in the proof of Theorem 2.4 to obtain a better estimate $c \text{dist}(P, K) \leq 6(C + 1)Q \text{dist}(P, K)$.

**Proof of Theorem 2.6.** Let $P \in M^{N \times n}$ be fixed and $P_{0} \in Q_{p}(K)$ be such that
\[
\text{dist}(P, Q_{p}(K)) = |P - P_{0}|.
\]

Since $P_{0} \in Q_{p}(K)$, by definition, $Q \text{dist}^{p}(P_{0}, Q_{p}(K)) = 0$. From Proposition 1.2, there exists a sequence $(\phi_{j})$ in $C_{0}^{\infty}(D_{n}, \mathbb{R}^{N})$, such that
\[
\lim_{j \to \infty} \int_{D_{n}} \text{dist}(P_{0} + D\phi_{j}(x), K) \, dx = 0, \tag{3.5}
\]

where $D_{n}$ is the unit cube in $\mathbb{R}^{n}$. Similar to the proof of Theorem 2.4, we have, because $D\phi_{j}$ is bounded for each fixed $j$ that we may apply Proposition 1.6 to find a sequence of measurable mappings $P_{j} : D_{n} \to K$, such that for each fixed $j$, $P_{j}$ is a bounded mapping, and
\[
\text{dist}^{p}(P_{0} + D\phi_{j}(x), K) = |P_{0} + D\phi_{j}(x) - P_{j}(x)|^{p}
\]

almost everywhere in $D_{n}$. Now, by the definition of quasiconvexification, for any given $\epsilon > 0$, we have

\[
Q \text{dist}^{p}(P, K) \leq \int_{D_{n}} Q \text{dist}^{p}(P + D\phi_{j}(x), K) \, dx
\]

\[
\leq \int_{D_{n}} \text{dist}^{p}(P + D\phi_{j}(x), K) \, dx
\]

\[
\leq \int_{D_{n}} |P + D\phi_{j}(x) - P_{j}(x)|^{p} \, dx
\]

\[
\leq (1 + \epsilon) \int_{D_{n}} |P - P_{0}|^{p} \, dx + C(\epsilon, p) \int_{D_{n}} |P_{0} + D\phi_{j}(x) - P_{j}(x)|^{p} \, dx
\]

\[= (1 + \epsilon) \text{dist}^{p}(P, Q_{p}(K)) + C(\epsilon, p) \int_{D_{n}} \text{dist}^{p}(P_{0} + D\phi_{j}(x), K) \, dx,
\]

where $C(\epsilon, p) > 0$ is a constant depending only on $\epsilon$ and $p$. Passing to the limit $j \to \infty$ in the above inequality, and taking into account of (3.5), we have
\[
Q \text{dist}^{p}(P, K) \leq (1 + \epsilon) \text{dist}^{p}(P, Q_{p}(K))
\]

for each fixed $\epsilon > 0$. Hence
\[
Q \text{dist}^{p}(P, K) \leq \text{dist}^{p}(P, Q_{p}(K))
\]
for every $P \in M^{N \times n}$. From Definition 1.1, we see that
\[ Q \operatorname{dist}^p(P, K) \leq Q \operatorname{dist}^p(P, Q_p(K)). \] (3.6)
From $K \subset Q_p(K)$, we always have
\[ Q \operatorname{dist}^p(P, Q_p(K)) \leq \operatorname{dist}^p(P, Q_p(K)) \leq \operatorname{dist}^p(P, K), \]
which again, by Definition 1.1, implies
\[ Q \operatorname{dist}^p(P, Q_p(K)) \leq Q \operatorname{dist}^p(P, K). \] (3.7)
Combining (3.6) and (3.7), the conclusion follows.

**Proof of Theorem 2.7.** We may assume that $|A| = 1$, $|B| = 1$. Then we may find $Q_0$, $Q \in SO(2)$, such that $A = Q_0E$, $B = QQ_0E_1Q^T = Q_0Q^2E_1$, where $E = \frac{1}{\sqrt{2}}I$, $I$ being the identity matrix, and $E_1$ is defined in Corollary 2.10. We seek to prove that $f(xQ_0E + yQQ_0E_1Q^T)$ is convex in $(x, y)$. Since $f$ is conjugating invariant and $RQ_0ER^T = Q_0E$ for all $R \in SO(2)$, we only need to prove that $f(Q_0[xE + yE_1])$ is convex. Since
\[ xE + yE_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} x + y & 0 \\ 0 & x - y \end{pmatrix}, \]
we let $x + y = u$, $x - y = v$ and define
\[ g(u, v) = f\left(\frac{1}{\sqrt{2}} Q_0 \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right). \]
Since $f$ is homogeneous of degree 1, so is $g$. $f$ is rank-1 convex, hence $g$ is separately convex. Apply Proposition 1.6, we see that $g$ is convex in $(u, v)$, so is $f(Q_0[xE + yE_1])$ in $(x, y)$. The proof is finished.

**Proof of Corollary 2.9.** Since $K$ is scaling invariant, we see that $\operatorname{dist}(\cdot, K)$ and $Q \operatorname{dist}(\cdot, K)$ are both homogeneous of degree 1. Because $Q \operatorname{dist}(\cdot, K)$ is also rank-1 convex and $Q \operatorname{dist}(\cdot, K) \geq \operatorname{dist}(\cdot, C(K))$, Theorem 2.7 implies that $Q \operatorname{dist}(\cdot, K)$ is a convex function on $S(A, B)$. Therefore, $Q_1(K) = C(K)$. Finally, since $\operatorname{dist}(\cdot, C(K))$ is a convex function, hence is quasiconvex. We then have, from Theorem 2.6 that
\[ Q \operatorname{dist}(\cdot, K) = Q \operatorname{dist}(\cdot, Q_1(K)) = Q \operatorname{dist}(\cdot, C(K)) = \operatorname{dist}(\cdot, C(K)). \]
Thus, $Q \operatorname{dist}(\cdot, K)$ is convex.

**Proof of Corollary 2.10.** Using a similar method as in the proof of Corollary 2.9, we see that $Q_1(C_P \cap \operatorname{span}[A, P]) = C(C_P \cap \operatorname{span}[A, P])$ for every $A \in E_5$. Since
\[ Q \operatorname{dist}(\cdot, C_P \cap \operatorname{span}[A, P]) \geq Q \operatorname{dist}(\cdot, C_P), \]
we have $C(C_P \cap \operatorname{span}[A, P]) \subset Q_1(C_P)$. Since we also have
\[ C(C_P) = \cup_{A \in E_5} C(C_P \cap \operatorname{span}[A, P]), \]
we see that $Q_1(C_P) = C(C_P)$. A similar argument as in the proof of Corollary 2.9 gives
\[ Q \operatorname{dist}(\cdot, C_P) = \operatorname{dist}(\cdot, C(C_P)). \]
The proof is complete.

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References


