Korovkin Sets and Mean Ergodic Theorems

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Korovkin-type theorems are established, and consequently mean ergodic theorems are obtained.

1. Introduction

Let $E$ be a normed linear space with its dual space $E^*$ and let $B[E]$ denote the normed algebra of all bounded linear operators of $E$ into itself with the identity operator $I$. Let $\mathcal{K}$ be a subset of $B[E]$ and let $T \in \mathcal{K}$. A subset $K$ of $E$ is said to be a $\mathcal{K}$-Korovkin set for $T$ if for any bounded sequence $\{T_n\}$ in $\mathcal{K}$, the relation

$$\lim_{n \to \infty} \|T_n(g) - T(g)\| = 0 \quad \text{for all } g \in K$$

implies that

$$\lim_{n \to \infty} \|T_n(f) - T(f)\| = 0 \quad \text{for all } f \in E.$$ 

Let $\mathcal{L}$ be a subset of $E^*$ and let $\mu \in \mathcal{L}$. A subset $K$ of $E$ is said to be an $\mathcal{L}$-Korovkin set for $\mu$ if for any bounded sequence $\{\mu_n\}$ in $\mathcal{L}$, the relation

$$\lim_{n \to \infty} \mu_n(g) = \mu(g) \quad \text{for all } g \in K$$

implies that

$$\lim_{n \to \infty} \mu_n(f) = \mu(f) \quad \text{for all } f \in E.$$ 

For the background of the Korovkin-type approximation theory, see the recent book of Altomare and Campiti [2], in which an excellent source and a vast literature of this theory can be found (cf. [3], [6], [7]).

The purpose of this paper lies in considering $\mathcal{K}$ and $\mathcal{L}$-Korovkin sets under certain requirements from a mean ergodic point of view. For the fundamental results about the ergodic theory, see [4; VIII] and for further extensive treatments of ergodic theorems, we refer to [8].

2. $\mathcal{K}$ and $\mathcal{L}$-Korovkin sets and mean ergodic theorems

If $S$ is a subset of $E$, then $S^\perp$ denotes the annihilator of $S$. That is,

$$S^\perp = \{\mu \in E^* : \mu(f) = 0 \quad \text{for all } f \in S\}.$$
If $\mathcal{L}$ is a subset of $E^*$, then we define
\[ \mathcal{L}_\perp = \{ f \in E : \mu(f) = 0 \quad \text{for all } \mu \in \mathcal{L} \}, \]
which is called the annihilator of $\mathcal{L}$. If $T$ is an operator in $B[E]$, then $\mathcal{R}_T$ denotes the range of $I - T$.

We shall need the following basic result.

**Lemma 2.1 (see [9; Theorem 4.6.1]).** If $S$ is a linear subspace of $E$, then $(S^\perp)_\perp$ coincides with the closure of $S$.

Let $\mu \in E^*$ and $T \in B[E]$. Then we say that $\mu$ is $T$-invariant if $\mu(T(f)) = \mu(f)$ for every $f \in E$, i.e., $\mu$ belongs to $\mathcal{R}_T^\perp$. Note that $\mu$ is $T$-invariant if and only if it is a fixed point of the adjoint operator $T^*$ of $T$, i.e., $T^*(\mu) = \mu$.

From now on, let $e$ be any fixed non-zero element in $E$, and we set
\[ \mathfrak{X} = \{ L \in B[E] : L(e) = e \}, \]
which is a closed convex subset of $B[E]$. Let $\varphi$ be an element in $E^*$ with $\varphi(e) = 1$, and we define
\[ P(f) = \varphi(f)e \quad \text{for all } f \in E. \quad (2.1) \]
Evidently, $P$ is a projection operator on $E$ belonging to $\mathfrak{X}$ and $\varphi$ is $P$-invariant.

Let $T, L \in B[E]$ and $n = 1, 2, 3, \ldots$. Then we define
\[ \sigma_{n,T} = \frac{1}{n} \sum_{i=0}^{n-1} T^i, \]
which is called the $n$-th Cesàro mean operator of $T$, and $T$ is said to be norm mean stable with $L$ if
\[ \lim_{n \to \infty} \| \sigma_{n,T}(f) - L(f) \| = 0 \quad \text{for all } f \in E. \quad (2.2) \]

The condition (2.2) implies that $L$ is necessarily a projection operator on $E$ and $TL = LT = L$. Furthermore, the mean ergodic theorem of Sine [16] (cf. [15]) asserts that if $E$ is a Banach space and if $\|T\| \leq 1$, then $T$ is norm mean stable with some $L \in B[E]$ if and only if the set of all fixed points of $T$ separates the set of all fixed points of $T^*$.

**Theorem 2.2.** Let $T \in B[E]$ and suppose that $\varphi$ is $T$-invariant.

(a) If the annihilator of $\mathcal{R}_T$ is spanned by $\varphi$, then $\mathcal{R}_T$ is a $\mathfrak{X}$-Korovkin set for $P$.

(b) If $T \in \mathfrak{X}$,
\[ \lim_{n \to \infty} \frac{\| T^n(f) \|}{n} = 0 \quad \text{for every } f \in E \quad (2.3) \]
and
\[ \sup_{n \geq 1} \| \sigma_{n,T} \| < \infty, \quad (2.4) \]

then the converse of (a) is also true.
Proof. (a) Let \( \{ L_n \} \) be a bounded sequence in \( \mathcal{X} \) such that for every \( g \in \mathcal{R}_T \), \( \lim_{n \to \infty} \| L_n(g) - P(g) \| = 0 \), which is equivalent to \( \lim_{n \to \infty} \| L_n(g) \| = 0 \) because of \( P(g) = 0 \). Let \( \epsilon > 0 \) and \( f \in \mathcal{E} \). Then, by Lemma 2.1, there exists an element \( h \in \mathcal{R}_T \) such that \( \| f - P(f) - h \| < \epsilon \). Since \( L_nP = P \) for all \( n \), we have

\[
\| L_n(f) - P(f) \| \leq \| L_n(f) - P(f) - L_n(h) \| + \| L_n(h) \|
\]

\[
\leq \| L_n \| \| f - P(f) - h \| + \| L_n(h) \| < \epsilon \| L_n \| + \| L_n(h) \|,
\]

and so \( \lim_{n \to \infty} \| L_n(f) - P(f) \| = 0 \) by virtue of \( \sup_n \| L_n \| < \infty \) and \( \lim_{n \to \infty} \| L_n(h) \| = 0 \). Therefore, \( \mathcal{R}_T \) is an \( \mathcal{X} \)-Korovkin set for \( P \).

(b) Suppose that \( T \in \mathcal{X} \), (2.3) and (2.4) hold. Then \( \{ \sigma_{n,T} \} \) is a bounded sequence in \( \mathcal{X} \) satisfying \( \lim_{n \to \infty} \| \sigma_{n,T}(f - T(f)) \| = 0 \) for all \( f \in \mathcal{E} \), since

\[
\sigma_{n,T}(I - T) = \frac{1}{n}(I - T^n) \quad (n = 1, 2, 3, \ldots).
\]  

(2.5)

Assume now that \( \mathcal{R}_T \) is a \( \mathcal{X} \)-Korovkin set for \( P \). Then we have that \( \lim_{n \to \infty} \| \sigma_{n,T}(f) - P(f) \| = 0 \) for every \( f \in \mathcal{E} \). Let \( \mu \) be an arbitrary element in \( \mathcal{R}_T \). Then for all \( f \in \mathcal{E} \), we have

\[
\lim_{n \to \infty} \mu(\sigma_{n,T}(f)) = \mu(P(f)) = \varphi(f)\mu(e),
\]

which implies \( \mu(f) = \mu(e)\varphi(f) \), since

\[
\mu(\sigma_{n,T}(f)) = \mu(f) \quad (n = 1, 2, 3, \ldots).
\]

Thus, \( \mathcal{R}_T^+ \) is spanned by \( \varphi \).

Remark 2.3. If \( T \) is power bounded, i.e., \( \sup_{n \geq 1} \| T^n \| < \infty \), then (2.3) and (2.4) automatically hold. Also, by (2.5), (2.2) implies (2.3).

As a consequence of Theorem 2.2, we have the following.

Corollary 2.4. Let \( T \) be an operator in \( \mathcal{X} \) satisfying (2.3), (2.4) and \( T^*(\varphi) = \varphi \). Then the following statements are equivalent:

(a) \( \mathcal{R}_T^+ \) is spanned by \( \varphi \).

(b) \( \mathcal{R}_T \) is a \( \mathcal{X} \)-Korovkin set for \( P \).

(c) \( T \) is norm mean stable with \( P \).

Let \( \mathcal{L} \) be a subset of \( E^* \) and \( \mu \in \mathcal{L} \). Then an operator \( T \in B[\mathcal{E}] \) is said to be \( \mathcal{L} \)-uniquely ergodic with \( \mu \) if \( \mu \) is only one \( T \)-invariant functional in \( \mathcal{L} \), or equivalently, \( T^* \) has exactly one fixed point \( \mu \) in \( \mathcal{L} \), i.e.,

\[
\{ \lambda \in \mathcal{L} : T^*(\lambda) = \lambda \} = \{ \mu \}.
\]

By [1; Corollary 1.2] and the theorem of Krein-Šmulian (see, [9; Theorem 10.2.1]), we have the following.

Remark 2.5. Suppose that \( \mathcal{E} \) is a separable Banach space and let \( \mathcal{L} \) be a convex subset of \( E^* \) such that the set

\[
\mathcal{L} \cap \{ \lambda \in E^* : \| \lambda \| \leq r \}
\]

is weak*-closed for each \( r > 0 \). Let \( T \in B[\mathcal{E}] \), and let \( \mu \) be a functional in \( \mathcal{L} \) which is \( T \)-invariant. Then \( T \) is \( \mathcal{L} \)-uniquely ergodic with \( \mu \) if and only if \( \mathcal{R}_T \) is an \( \mathcal{L} \)-Korovkin set for \( \mu \).
3. Korovkin sets and mean ergodic theorems in function spaces

In this section, let $E$ be a function space on a non-empty set $X$. That is, $E$ is a normed linear space of real or complex valued functions on $X$, which contains the unit function $e$ defined by $e(x) = 1$ for all $x \in X$. Consequently, all the results obtained in the preceding section are applicable to this setting.

From now on, let $X$ be a compact metric space and let $C(X)$ denote the Banach space of all real valued continuous functions on $X$ with the usual supremum norm. Note that $C(X)$ is separable. Let $E$ be a linear subspace containing the unit function $e$. For a point $x \in X$, we define the point evaluation functional $\delta_x$ at $x$ by $\delta_x(f) = f(x)$ for all $f \in E$.

If $\mathcal{L}$ is a subset of $E^*$, then $\mathcal{F}(\mathcal{L})$ denotes the set of all operators $L \in B[E]$ such that $\delta_x \circ L$ belongs to $\mathcal{L}$ for every $x \in X$. Set

$$\mathcal{L}^1 = \{ \mu \in E^* : \mu(e) = 1 \}$$

and

$$\mathcal{F}^1 = \{ L \in B[E] : L(e) = e \}.$$ 

Then we have $\mathcal{F}(\mathcal{L}^1) = \mathcal{F}^1$. Let $\mathcal{L}_+^1$ denote the set of all positive linear functionals on $E$, and we put $\mathcal{F}_+^1 = \mathcal{F}(\mathcal{L}_+^1)$, which consists of all positive linear operators of $E$ into itself. Furthermore, we set $\mathcal{L}_+^1 = \mathcal{L}_+ \cap \mathcal{L}^1$ and $\mathcal{F}_+^1 = \mathcal{F}(\mathcal{L}_+^1)$, which coincides with $\mathcal{F}_+ \cap \mathcal{F}^1$.

Recall that $\varphi \in \mathcal{L}^1$ and $P$ is the projection operator in $\mathcal{F}^1$ defined by (2.1).

**Theorem 3.1.** Let $T \in \mathcal{L}_+^1$. Suppose that $\varphi \in \mathcal{L}_+^1$ and $T^*(\varphi) = \varphi$. Then $\mathcal{R}_T$ is a $\mathcal{F}_+^1$-Korovkin set for $P$ if and only if $T$ is norm mean stable with $P$.

**Proof.** Note that $\{ \sigma_{n,T} \}$ is a bounded sequence in $\mathcal{F}_+^1$ with $\| \sigma_{n,T} \| = 1$ for all $n = 1, 2, 3, \cdots$. Since $\| T \| = 1$ and $P$ vanishes on $\mathcal{R}_T$, (2.5) yields that $\lim_{n \to \infty} \| \sigma_{n,T}(g) - P(g) \| = 0$ for all $g \in \mathcal{R}_T$. Therefore, if $\mathcal{R}_T$ is a $\mathcal{F}_+^1$-Korovkin set for $P$, then $T$ is norm mean stable with $P$.

Conversely, suppose that $T$ is norm mean stable with $P$. Let $\lambda$ be any functional in $\mathcal{L}_+^1$ with $T^*(\lambda) = \lambda$. Then we are able to extend $\lambda$ to a positive linear functional $\nu$ on the whole space $C(X)$. By the Riesz representation theorem, there exists a probability measure $\rho$ on $X$ such that

$$\nu(f) = \int_X f(x) \, d\rho(x) \quad \text{for all } f \in C(X).$$

Let $g$ be an arbitrary function in $E$. Then we have

$$|\sigma_{n,T}(g)(x)| \leq \| \sigma_{n,T} \| \| g \|$$

for all $x \in X$ and for each $n = 1, 2, 3, \cdots$. Therefore, it follows that

$$\varphi(g) = \int_X P(g)(x) \, d\rho(x) = \lim_{n \to \infty} \int_X \sigma_{n,T}(g)(x) \, d\rho(x)$$

$$= \lim_{n \to \infty} \nu(\sigma_{n,T}(g)) = \lim_{n \to \infty} \lambda(\sigma_{n,T}(g)) = \lambda(g).$$

Thus we have $\lambda = \varphi$, and so it follows from [5; Theorems 1.1 and 1.2] that $\mathcal{R}_T$ is a $\mathcal{F}_+^1$-Korovkin set for $P$. \qed
Remark 3.2. Let $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ be a finite set of continuous mappings from $X$ into itself and $F = \{f_1, f_2, \cdots, f_m\}$ a finite subset of $E$. We define

$$T_{\alpha,F}(f) = \sum_{i=1}^{m} (f \circ \alpha_i) f_i$$

for all $f \in E$. Then $T_{\alpha,F}$ is a bounded linear operator of $E$ into $C(X)$. Assume that $T_{\alpha,F}$ maps $E$ into itself. Then all the results presented in this section are applicable to $T = T_{\alpha,F}$.

Finally, in view of the study of the rate of convergence for approximation processes of positive linear operators, we notice that our forthcoming topic is to give a quantitative version of Theorem 3.1, with an optimal order of approximation (cf. [10], [11], [12], [13], [14]).

References