MOTION OF CHARGED PARTICLES FROM THE GEOMETRIC VIEW POINT

OSAMU IKAWA

Communicated by Charles-Michel Marle

Abstract. This is a review article on the motion of charged particles related to the author’s study. The equation of motion of a charged particle is defined as a curve satisfying a certain differential equation of second order in a semi-Riemannian manifold furnished with a closed two-form. Charged particle is a generalization of geodesic. We shall oversee the geometric aspect of charged particles.

1. Introduction

Let $F$ be a closed two-form and $U$ a function on a connected semi-Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is a semi-Riemannian metric on $M$. We denote by $\bigwedge^m(M)$ the space of $m$-forms on $M$. Denote by $\iota(X): \bigwedge^m(M) \to \bigwedge^{m-1}(M)$ the interior product operator induced from a vector field $X$ on $M$, and by $\mathcal{L}: T(M) \to T^*(M)$, the Legendre transformation from the tangent bundle $T(M)$ of $M$ onto the cotangent bundle $T^*(M)$, which is defined by

$$\mathcal{L}: T(M) \to T^*(M), \quad u \mapsto \mathcal{L}(u), \quad \mathcal{L}(u)(v) = \langle u, v \rangle, \quad u, v \in T(M). \quad (1)$$

A curve $x(t)$ in $M$ is called the motion of a charged particle under electromagnetic field $F$ and potential energy $U$, if it satisfies the following second order differential equation

$$\nabla_\dot{x} \ddot{x} = -\text{grad} U - \mathcal{L}^{-1}(\iota(\dot{x})F) \quad (2)$$

where $\nabla$ is the Levi-Civita connection of $M$. Here $\nabla_\dot{x} \ddot{x}$ means the acceleration of the charged particle. Since $-\mathcal{L}^{-1}(\iota(\dot{x})F)$ is perpendicular to the direction $\dot{x}$ of the movement, $-\mathcal{L}^{-1}(\iota(\dot{x})F)$ means the Lorentz force. This equation originated in the theory of general relativity (see § 2 or [26]). When $F = 0$ and $U = 0$, then $x(t)$ is merely a geodesic. When $M$ is a Kähler manifold with a complex structure $J$, then it is natural to take a scalar multiple of the Kähler form $\Omega$ defined