ON HENRI POINCARÉ’S NOTE “Sur une forme nouvelle des équations de la Mécanique”

CHARLES-MICHEL MARLE *

Communicated by Ivaïlo M. Mladenov

Abstract. We present in modern language the contents of the famous note published by Henri Poincaré in 1901, in which he proves that, when a Lie algebra acts locally transitively on the configuration space of a Lagrangian mechanical system, the well known Euler-Lagrange equations are equivalent to a new system of differential equations defined on the product of the configuration space with the Lie algebra. We write these equations, called the Euler-Poincaré equations, in an intrinsic form, without any reference to a particular system of local coordinates, and prove that they can be conveniently expressed in terms of the Legendre and momentum maps. We discuss the use of the Euler-Poincaré equation for reduction (a procedure sometimes called Lagrangian reduction by modern authors), and compare this procedure with the well known Hamiltonian reduction procedure (formulated in modern terms in 1974 by J. Marsden and A. Weinstein). We explain how a break of the symmetry in the phase space produces the appearance of a semi-direct product of Lie groups.

Contents

1 Introduction 2

2 Poincaré’s Note 4

2.1 Derivation of the Euler-Poincaré equation 4

2.2 Comments Made by Poincaré 10

3 Two Special Cases of the Euler-Poincaré Equation 11

3.1 Euler-Lagrange Equation 11

3.2 Euler Equation for the Motion of a Rigid Body 12

4 The Euler-Poincaré Equation in Terms of the Legendre and the Momentum Maps 14

4.1 The Lift to $T^*Q$ of the Lie Algebra Action $\psi$ and the Momentum Map 14

4.2 The Legendre Map $L$ 15

* In memory of Jean-Marie Souriau, founder of the modern theory of Geometrical Mechanics, with respect and admiration.

doi: 10.7546/jgsp-29-2013-1-38