The Automorphisms of Generalized Witt Type Lie Algebras

Naoki Kawamoto, Atsushi Mitsukawa, Ki-Bong Nam, and Moon-Ok Wang

Communicated by E. Vinberg

Abstract. We find the Lie automorphisms of generalized Witt type Lie algebras $W[x, e^x]$ and $W[x, e^{\pm x}]$.

1. Introduction

Simplicity of several generalized Witt type Lie algebras have been considered by many authors over a field $F$ of characteristic zero. Kac [3] studied the generalized Witt algebra on the $F$-algebra in the formal power series algebra $F[[x_1, \ldots, x_n]]$. There exist many generalized Witt type simple Lie algebras using the algebras stable under the action of derivations ([1], [3], [4], [6]). We consider one-variable cases based on using the exponential functions. Let $\partial = \frac{d}{dx}$, $F[x^\pm 1, e^{\pm x}]$, and let $F[a_1, \ldots, a_n]$ be a subalgebra of $F[x^\pm 1, e^{\pm x}]$ generated by $a_1, \ldots, a_n$. If $F[a_1, \ldots, a_n]$ is $\partial$-stable we put $W[a_1, \ldots, a_n] = \{f \partial \mid f \in F[a_1, \ldots, a_n]\}$. Then $W[a_1, \ldots, a_n]$ is a Lie algebra over $F$ with the usual product

\[ [f \partial, g \partial] = (f \partial \circ g \partial - g \partial \circ f \partial = (f(g \partial) - (\partial f)g) \partial \quad (f, g \in F[a_1, \ldots, a_n]). \]

The Lie algebras $W[x]$, $W[x^\pm 1]$, $W[e^{\pm x}]$, $W[x, e^{\pm x}]$, and $W[x^\pm 1, e^{\pm x}]$ are simple, while $W[x, e^x]$ and $W[x^\pm 1, e^x]$ are not simple. The automorphisms of $W[x]$ is considered in [7] (cf. also [2]). The automorphisms of generalized Witt type Lie algebras of Laurent polynomials are considered in [1], [5]. In this paper we find the Lie automorphisms of $W[x, e^x]$ and $W[x, e^{\pm x}]$ containing polynomials and exponential functions. The automorphism group of $W[x, e^x]$ is isomorphic to $F^* \times F$, while the automorphism group of $W[x, e^{\pm x}]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \rtimes (F^* \times F)$.

2. Preliminaries

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_+$ the set of positive integers, $\mathbb{Z}_-$ the set of negative integers, and $\mathbb{N}$ the set of non-negative integers. For the field $F$ we denote by...
Let \( \alpha, \beta \) be non-zero elements of \( W[x, e^x] \) such that \( [\alpha, \beta] = \beta \). Then \( \alpha - \frac{1}{n-1}(x+c)\partial, \beta \in (x+c)^n\partial \) for some \( c \in F \) and \( n \in \mathbb{N} \setminus \{1\} \).

**Proof.** Let \( \alpha = f\partial, \beta = g\partial \) and let \( f = a_m x^m + \cdots + a_0, \ g = b_n x^n + \cdots + b_0 \), where \( m, n \geq 0 \) and \( a_m, b_n \neq 0 \). If \( m \neq n \), then from \( [\alpha, \beta] = \beta \) we have \( m = 1 \) and

\[
f = \frac{1}{n-1}(x+c), \quad g = b_n(x+c)^n
\]

for some \( c \in F \). If \( m = n \), then it follows by taking \( h = f-ag, \) where \( a = \frac{a_m}{b_n} \neq 0 \), that \( f = a_n(x+c)^n + \frac{1}{n-1}(x+c), \ g = b_n(x+c)^n \).

**3. Stabilizers**

We determine the elements \( \alpha, \beta \) satisfying the condition \([\alpha, \beta] = \beta \) in some generalized Witt type Lie algebras.

**Proposition 3.1.** Let \( \alpha, \beta \) be non-zero elements of \( W[x] \) such that \([\alpha, \beta] = \beta \). Then \( \alpha - \frac{1}{n-1}(x+c)\partial, \beta \in (x+c)^n\partial \) for some \( c \in F \) and \( n \in \mathbb{N} \setminus \{1\} \).

**Proof.** Let \( \alpha = f\partial, \beta = g\partial \) and let \( f = a_m x^m + \cdots + a_0, \ g = b_n x^n + \cdots + b_0 \), where \( m, n \geq 0 \) and \( a_m, b_n \neq 0 \). If \( m \neq n \), then from \([\alpha, \beta] = \beta \) we have \( m = 1 \) and

\[
f = \frac{1}{n-1}(x+c), \quad g = b_n(x+c)^n
\]

for some \( c \in F \). If \( m = n \), then it follows by taking \( h = f-ag, \) where \( a = \frac{a_m}{b_n} \neq 0 \), that \( f = a_n(x+c)^n + \frac{1}{n-1}(x+c), \ g = b_n(x+c)^n \).
Proposition 3.2. Let $\alpha, \beta$ be non-zero elements of $W[x, e^x]$ and $[\alpha, \beta] = \beta$. Then $\beta_+ = 0$ or $\beta_0 = 0$ and one of the following statements holds: (1) $\alpha - \frac{1}{n-1}(x + c)\partial, \beta \in \langle (x + c)^n \partial \rangle$ for some $c \in F$ and $n \in \mathbb{N} \setminus \{1\}$, or (2) $\alpha - \frac{1}{n} \partial, \beta \in \langle e^{nx} \partial \rangle$ for some $n \in \mathbb{Z}_+$.

Proof. Let $\alpha = (f_n e^{mx} + \cdots + f_0)\partial, \beta = (g_n e^{nx} + \cdots + g_0)\partial$, where $f_m, \ldots, f_0, g_n, \ldots, g_0 \in F[x], f_m, g_n \neq 0$, and $m, n \in \mathbb{N}$. If $m \neq n$, then by some computation we deduce from $[\alpha, \beta] = \beta$ that $m = 0$, $n > 0$ and that

$$\alpha = \frac{1}{n} \partial, \quad \beta = b_n e^{nx} \partial \quad (b_n \neq 0).$$

Let $m = n$. If $n = 0$, then we can apply Proposition 3.1. If $n > 0$, then we have $f_n g_n - f'_n g_n = 0$, $(\frac{f_n}{g_n})' = 0$, and $g_n = c f_n$ for some constant $c \neq 0$, where we write simply $f'$ instead of $\partial f$. From $[\alpha - \frac{1}{n} \beta, \beta] = \beta$ we have $\alpha = a e^{nx} \partial + \frac{1}{n} \partial, \beta = b e^{nx} \partial$ for some $a, b \in F$.

We continue to characterize the elements $\alpha, \beta$ satisfying the condition $[\alpha, \beta] = \beta$ in $W[x, e^{\pm x}]$.

Lemma 3.3. Let $\alpha, \beta$ be non-zero elements of $W[x, e^{\pm x}]$ and $[\alpha, \beta] = \beta$. Then

1. For $\overline{\alpha}$ and $\overline{\beta}$ we have either
   (i) $\alpha - k\beta - \frac{1}{n-1}(x + c)\partial, \beta \in \langle (x + c)^n \partial \rangle$ for some $k, c \in F, n \in \mathbb{N} \setminus \{1\}$, or
   (ii) $\alpha - k\beta = \frac{1}{n} \partial$ and $\beta \in \langle e^{nx} \partial \rangle$ for some $k \in F, n \in \mathbb{Z} \setminus \{0\}$.

2. For $\alpha$ and $\beta$ we have either
   (i) $\alpha - l\beta = \frac{1}{n} \partial, \beta \in \langle (x + d)^m \partial \rangle$ for some $l, d \in F, m \in \mathbb{N} \setminus \{1\}$, or
   (ii) $\alpha - l\beta = \frac{1}{n} \partial$ and $\beta \in \langle e^{nx} \partial \rangle$ for some $l \in F, m \in \mathbb{Z} \setminus \{0\}$.

Proof. We show Case (1), since Case (2) will be proved similarly. Since $[\alpha - k\beta, \beta] = \beta$ for any $k \in F$, if necessary we can replace $\alpha$ with $\alpha - k\beta$. Hence we may assume $\langle \overline{\alpha} \rangle \neq \langle \overline{\beta} \rangle$. Then by Note 2.2 we have $\langle \overline{\alpha}, \overline{\beta} \rangle \neq 0$. Therefore $[\overline{\alpha}, \overline{\beta}] = \overline{\beta}$ and $\overline{\alpha} \in W_0 = W[x]$ since $W[x, e^{\pm x}]$ is $\mathbb{Z}$-graded. We determine $\overline{\alpha}$ and $\overline{\beta}$. Apply the automorphism $\tau$ if necessary. Then by Proposition 3.2 we have $\overline{\alpha} - \frac{1}{n-1}(x + c)\partial, \beta \in \langle (x + c)^n \partial \rangle$ for some $c \in F, n \in \mathbb{N} \setminus \{1\}$, or $\overline{\alpha} - \frac{1}{n} \partial, \overline{\beta} \in \langle e^{nx} \partial \rangle$ for some $n \in \mathbb{Z} \setminus \{0\}$. In the later case $\overline{\alpha} = \frac{1}{n} \partial + be^{nx} \partial$ for some $b \in F$, and $\overline{\alpha} = \frac{1}{n} \partial$ since $\overline{\alpha}$ is homogeneous.

Lemma 3.4. Let $\alpha, \beta$ be non-zero elements of $W[x, e^{\pm x}]$ and $[\alpha, \beta] = \beta$. Then we have the following statements:

1. If $\alpha_+ \neq 0$, then $\beta_+ \neq 0, \langle \overline{\alpha} \rangle = \langle \overline{\beta} \rangle \subseteq W_n$ for some $n \in \mathbb{Z}_+$, and also $\beta = \frac{1}{k} \alpha_+ + \frac{1}{k} (\alpha_0 - \frac{1}{n} \partial) + \beta_-$ for some $k \in F^*$.

2. If $\alpha_- \neq 0$, then $\beta_- \neq 0, \langle \overline{\alpha} \rangle = \langle \overline{\beta} \rangle \subseteq W_m$ for some $m \in \mathbb{Z}_-$, and $\beta = \beta_+ + \frac{1}{l} (\alpha_0 - \frac{1}{m} \partial) + \frac{1}{l} \alpha_-$ for some $l \in F^*$.

3. If $\alpha_+ + \alpha_- \neq 0$, then $\beta \in \text{sp}\{\alpha_+, \alpha_-, \alpha_0, \partial\}$. 
Proof. Let $\alpha_+ \neq 0$. Then $\overline{\alpha} \in W_n$ for some $n \in \mathbb{Z}_+$. Assume that $[\overline{\alpha}, \overline{\beta}] \neq 0$. Then $[\overline{\alpha}, \overline{\beta}] = \overline{\beta}$. If $\beta \in W_m$, then $\overline{\beta} = [\overline{\alpha}, \overline{\beta}] \in W_{n+m}$, a contradiction. Hence $[\overline{\alpha}, \overline{\beta}] = 0$, and by Note 2.2 we have $\langle \overline{\alpha} \rangle = \langle \overline{\beta} \rangle$ and $\beta_+ \neq 0$. Hence $\overline{\beta} \in W_n$, and we have $\overline{\alpha} - k\overline{\beta} = \overline{\frac{k}{n}\partial}$ for some non-zero $k \in F$ by Lemma 3.3. Then $\alpha - k\beta = \frac{1}{n}\partial + \alpha_+ - k\beta_+$ and $\beta = \frac{1}{k}\alpha_+ + \frac{1}{k} \left( \alpha_0 - \frac{1}{n}\partial \right) + \beta_-$.

(2) Let $\alpha_- \neq 0$. Then $\langle \alpha \rangle \in W_m$ for some $m \in \mathbb{Z}_-$, and we have $\beta = \beta_+ + \frac{1}{k} \left( \alpha_0 - \frac{1}{m}\partial \right) + \frac{1}{l}\alpha_-$ for some $l \in F^*$. 

(3) Let $\alpha_+, \alpha_- \neq 0$. Then from (1) and (2) we have $
abla = \frac{1}{k} \alpha_+ + \frac{1}{k} \left( \alpha_0 - \frac{1}{n}\partial \right) + \beta_-$ for some $k, l \in F^*$, $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}_-$. Thus $\beta \in \text{sp}\{\alpha_+, \alpha_-, \alpha_0, \partial\}$. 

Lemma 3.5. Let $\alpha$ be a non-zero element of $W[x, e^{\pm x}]$, and let $\{\beta_i \mid i \in I\}$ be an infinite and linearly independent subset of $W[x, e^{\pm x}]$. If $[\alpha, \beta_i] = a_i \beta_i$ and $a_i \neq 0$ for any $i \in I$, then $\alpha_0 \neq 0$ and either $\alpha_+ = 0$ or $\alpha_- = 0$.

Proof. Assume that $\alpha_+ \neq 0$ and $\alpha_- \neq 0$. Since $\left[ \frac{1}{a_i}, \alpha, \beta_i \right] = \beta_i$, by Lemma 3.4(3) the set $\{\beta_i \mid i \in I\}$ is contained in the finite dimensional subspace $\text{sp}\{\alpha_+, \alpha_-, \alpha_0, \partial\}$, a contradiction. Hence $\alpha_+ = 0$ or $\alpha_- = 0$. If both $\alpha_+ = 0$ and $\alpha_- = 0$, then clearly $\alpha_0 \neq 0$. Let $\beta = \beta_i$. If $\alpha_+ \neq 0$, then we apply the automorphism $\tau$. Hence we may assume that $\alpha = \alpha_+ + \alpha_0$. By Lemma 3.4 we have

$$\beta = \frac{1}{ka_i} \alpha_+ + \frac{1}{k} \left( \frac{1}{a_i} \alpha_0 - \frac{1}{n}\partial \right) + \beta_-$$

for some $k \in F^*$ and $n \in \mathbb{Z}_+$ such that $\overline{\beta} \in W_n$. Hence $\beta_+ \neq 0$. If $\beta_- = 0$, then by Proposition 3.2 we have $\frac{1}{a_i} \alpha_0 - \frac{1}{n}\partial = k\beta_0 = 0$ and $\alpha_0 \neq 0$. If $\beta_- \neq 0$, then $\left[ \frac{1}{a_i}, \alpha, \beta \right] = \overline{\beta}$ since $\langle \alpha \rangle \neq \langle \beta \rangle$. Hence $\alpha_0 = \alpha \neq 0$.

4. Automorphisms

We determine the automorphisms of $W[x, e^x]$ and $W[x, e^{\pm x}]$ in this section.

Lemma 4.1. Let $\varphi$ be an injective homomorphism of $W[x]$. Then $\varphi(x^n \partial) = a^{n-1}(x + b)^n \partial$ ($n \in \mathbb{N}$) for some $a \in F^*$, $b \in F$.

Proof. Let $\varphi$ be an injective homomorphism of $W[x]$. Since

$$\langle \varphi(x^m \partial), \varphi(x^n \partial) \rangle = (n - m) \varphi(x^{m+n-1} \partial),$$

we have $\left[ \frac{1}{n-m} \varphi(x \partial), \varphi(x^n \partial) \right] = \varphi(x^n \partial)$ ($n \neq 1$). Since $\varphi(x^n \partial)$ ($n \in \mathbb{N}$) are linearly independent it follows easily from Proposition 3.1 that \n
$$\varphi(x^n \partial) = a_{n-1}(x + b)^n \partial \quad (n \in \mathbb{N})$$

for some $a_{n-1} \in F^*$, $b \in F$. Then from (2) we have $a_{m-1}a_{n-1} = a_{m+n-2} = a_{m-1+n-1}$ ($n, m \in \mathbb{N}$, $n \neq m$), that is, $a_na_n = a_{n+m}$ ($n, m \in \mathbb{N} \cup \{-1\}$, $n \neq m$). By Note 2.4, $a_n = a_1^n$ and $\varphi(x^n \partial) = a^{n-1}(x + b)^n \partial$, where $a = a_1 \in F^*$. 

$\blacksquare$
Lemma 4.1 we have

Proof. Let

The automorphism group of

Theorem 4.5. This is clear from

Proof. The automorphism group of

Corollary 4.4. Let \( \varphi \) be an automorphism of \( W[x,e^z] \). Then \( \varphi(W[x]) \subseteq W[x] \).

Proof. It holds that \( [\varphi(x\partial), \varphi(x^n\partial)] = (n-1)\varphi(x^n\partial) \) \((n \in \mathbb{N})\). Let \( \alpha = \varphi(x\partial) \). Then by Lemma 3.5 we have \( \alpha_0 \neq 0 \), and \( \alpha_+ = 0 \) or \( \alpha_- = 0 \). Let \( \beta = \varphi(\partial) \). Then similarly from \( [\varphi(\partial), \varphi(e^{mx}\partial)] = m\varphi(e^{mx}\partial) \) \((m \in \mathbb{N})\) we have \( \beta_0 \neq 0 \), and \( \beta_+ = 0 \) or \( \beta_- = 0 \). Assume that \( \alpha_+ 
eq 0 \). Then from \([-\alpha, \beta] = \beta \) and Lemma 3.4(1) we have \( \beta_+ \neq 0 \) and \( \alpha, \beta \in W[x,e^z] \). Hence \( \beta_0 \neq 0 \) and \( \beta_+ \neq 0 \), but this contradicts to Proposition 3.2. Assume that \( \alpha_+ \neq 0 \). Then applying \( \tau \) we have a contradiction similar to the above. Therefore \( \alpha = \alpha_0 = 0 \) in \( W[x] \). Then the case \( \beta_+ \neq 0 \) and the case \( \beta_- \neq 0 \) cause similar contradictions. Thus \( \beta = \beta_0 \in W[x] \). From \([\beta, \varphi(x^n\partial)] = (n-1)\varphi(x^{n-1}\partial) \) we have \( \varphi(x^n\partial) \in W[x] \) \((n \in \mathbb{N})\) by induction.

Theorem 4.3. Let \( \varphi \) be an automorphism of \( W[x,e^z] \). Then \( \varphi \) is a product of \( \varphi_a \) and \( \psi_b \) for some \( a \in F^*, b \in F \).

Proof. Let \( \varphi \) be an automorphism of \( W[x,e^z] \). Then by Proposition 4.2 and Lemma 4.1 we have \( \varphi(\partial) = a^{-1}\partial \) and \( \varphi(x^n\partial) = a^{n-1}(x+b)^n\partial \) for some \( a \in F^*, b \in F \). Since \([\varphi(\partial), \varphi(e^{mx}\partial)] = m\varphi(e^{mx}\partial)\), we have \([\partial, \varphi(e^{mx}\partial)] = am\varphi(e^{mx}\partial)\). Then by Note 2.3 we have \( \varphi(e^{mx}\partial) \in \langle e^{amx}\partial \rangle \) and \( am \in \mathbb{N} \). Since \( \varphi \) is surjective, it follows that \( a = 1 \), \( \varphi(x^n\partial) = (x+b)^n\partial \) \((n \in \mathbb{N})\), and \( \varphi(e^{mx}\partial) = c_m e^{mx}\partial \) \((m \in \mathbb{N})\). Then from \([\varphi(e^{mx}\partial), \varphi(e^{kx}\partial)] = (k-m)\varphi(e^{(m+k)x}\partial) \) \((m, k \in \mathbb{N})\) and Note 2.4, we have \( c_m = c^m \) for some \( c \in F^* \). Thus \( \varphi(e^{mx}\partial) = c^m e^{mx}\partial \). Hence \((\varphi_c \circ \psi_b)^{-1} \circ \varphi = 1_{W[x,e^z]} \) by Note 2.1, and therefore \( \varphi = \varphi_c \circ \psi_b \).

Corollary 4.4. The automorphism group of \( W[x,e^z] \) is isomorphic to \( F^* \times F \).

Proof. This is clear from \( \varphi_a \circ \psi_b = \psi_b \circ \varphi_a \) for any \( a \in F^*, b \in F \).

Theorem 4.5. An automorphism of \( W[x,e^{\pm x}] \) is a product of \( \varphi_a, \psi_b, \) and \( \tau \) for some \( a \in F^*, b \in F \).

Proof. Let \( \varphi \) be an automorphism of \( W[x,e^{\pm x}] \). Then as in the proof of Theorem 4.3 \( \varphi(x^n\partial) = a^{n-1}(x+b)^n\partial \) \((n \in \mathbb{N})\) and \( \varphi(e^{mx}\partial) = c_m e^{amx}\partial \) \((m \in \mathbb{Z})\). Since \( \varphi \) is surjective, we have \( a = \pm 1 \), and applying \( \tau \) if necessary we may assume \( a = 1 \). Then it follows that \( \varphi(e^{mx}\partial) = c^m e^{mx}\partial \) for some \( c \in F^* \) and that \((\varphi_c \circ \psi_b)^{-1} \circ \varphi = 1_{W[x,e^{\pm x}]} \).

Corollary 4.6. The automorphism group of \( W[x,e^{\pm x}] \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times (F^* \times F) \).

Proof. This is clear from \( \varphi_a \circ \psi_b = \psi_b \circ \varphi_a, \tau \circ \varphi_a \circ \tau = \varphi_a^{-1}, \) and \( \tau \circ \psi_b \circ \tau = \psi_{-b} \).
References


Naoki Kawamoto
Japan Coast Guard Academy
5-1 Wakaba
Kure 737-8512
Japan
kawamoto@jcga.ac.jp

Atsushi Mitsukawa
Department of Management Information
Fukuyama Heisei University
117-1 Miyuki
Fukuyama 720-0001
Japan
mitukawa@fuhc.fukuyama-u.ac.jp

Ki-Bong Nam
Department of Mathematics and
Computer Science
University of Wisconsin-Whitewater
800 West Main Street
Whitewater
WI 53190
USA
namk@mail.uww.edu

Moon-Ok Wang
Department of Mathematics
Hanyang University
Ansan
Kyunggi 425-791
Korea
wang@mail.hanyang.ac.kr

Received September 28, 2002
and in final form March 3, 2003