We construct, via a complex $G$–bundle space, a Weil representation for the group $G = SL_*(2, \mathbb{A})$, where $(\mathbb{A}, \ast)$ is a locally profinite ring with involution. The construction is obtained using maximal isotropic lattices and Heisenberg groups.

1. Preliminaries.

Let $(\mathbb{A}, \ast)$ be a locally profinite ring with involution, i.e. a unitary locally compact and totally disconnected ring with an involutive anti-automorphism $a \mapsto a^\ast$, $a \in \mathbb{A}$. Let $Z_s(\mathbb{A})$ be the subring of central symmetric elements of $\mathbb{A}$.

We define the group $GL_*(2, \mathbb{A})$ of matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{A}$, such that:

1. $ab^\ast = ba^\ast$, $cd^\ast = dc^\ast$
2. $a^\ast c = c^\ast a$, $b^\ast d = d^\ast b$
3. $ad^\ast - bc^\ast = a^\ast d - c^\ast b$ is an invertible central symmetric element of $\mathbb{A}$, i.e. an element of $Z_s(\mathbb{A})^\times$.

We set $det_\ast (g) = ad^\ast - bc^\ast = a^\ast d - c^\ast b$; then

$$g^{-1} = [det_\ast (g)]^{-1} \begin{pmatrix} d^\ast & -b^\ast \\ -c^\ast & a^\ast \end{pmatrix}$$

We observe that the function $det_\ast : GL_*(2, \mathbb{A}) \rightarrow Z_s(\mathbb{A})^\times$ is an epimorphism so that $G = SL_*(2, \mathbb{A}) = \text{Ker } det_\ast$ is a normal subgroup of $GL_*(2, \mathbb{A})$. 

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In what follows we will assume that $Z_q(A) = F$ is a $p$-adic field. We denote by $O_F$ the ring of integers of $F$, $P_F$ is the maximal ideal of $O_F$, $\varpi$ is a generator of $P_F$ and $k_F$ is the residual field of $F$ which has $q$ elements.

Some such rings are: $A = M_n(F)$, $F$ a $p$-adic field, with $*$ the transposition; $A = K$ a separable quadratic extension of $F$, $F$ as above with $*$ the non trivial Galois element; $A = \mathbb{A}^0 \mathbb{V} \oplus \mathbb{A}^1 \mathbb{V} \oplus \mathbb{A}^2 \mathbb{V}$ where $V$ is a two dimensional vector space over a $p$-adic field $F$ with basis $(e_1, e_2)$ and $*$ is given by the basis transposition $(e_1, e_2)$ to $(e_2, e_1)$.

2. General Setting

Let $H$ be a locally profinite group and $\Gamma$ a subgroup of $\text{Aut}(H)$. Let $(\pi, V)$ be an irreducible smooth (complex) representation of $H$ such that $\pi^\gamma \simeq \pi$ ($\pi^\gamma = \pi \circ \gamma$) for every $\gamma$ in $\Gamma$.

If $\gamma \in \Gamma$ then there exists $T_\gamma \in \text{Aut}(C(V))$ such that $T_\gamma \pi(x) = \pi \gamma(x) T_\gamma$ for every $x \in H$.

Set $G$ be the semidirect product of $\Gamma$ and $H$. For $(\gamma, h)$ in $G$ we define $\tilde{\pi}(\gamma, h)$ in $\text{Aut}(C(V))$ by

\[ \tilde{\pi}(\gamma, h) = T_\gamma \pi(h). \]

**Proposition 2.1.** The endomorphism $\tilde{\pi}$, defined above, is a projective extension of $\pi$ to $G$.

**Proof.** We want to prove that $T_{\gamma \delta}^{-1} T_{\gamma} T_{\delta}$ is a scalar.

Since $T_{\gamma} T_{\delta} \pi(x) = T_{\gamma} \pi(\delta(x)) T_{\delta} = \pi(\gamma \delta(x)) T_{\gamma} T_{\delta}$ and $T_{\gamma \delta} \pi(x) = \pi(\gamma \delta(x)) T_{\gamma \delta}$ then

\[ T_{\gamma \delta}^{-1} T_{\gamma} T_{\delta} \pi(x) = \pi(x) T_{\gamma \delta}^{-1} T_{\gamma} T_{\delta}. \]

It follows, by Schur’s Lemma, that $T_{\gamma \delta}^{-1} T_{\gamma} T_{\delta} = \sigma(\gamma, \delta) id_V$, for a cocycle $\sigma$.

We compute now $\tilde{\pi}(\gamma, h) \tilde{\pi}(\delta, k)$. We have

\[ \tilde{\pi}(\gamma, h) \tilde{\pi}(\delta, k) = \sigma(\gamma, \delta) T_{\gamma \delta} \pi(\delta^{-1}(h)) \pi(k). \]

Since $\tilde{\pi}((\gamma, h)(\delta, k)) = \tilde{\pi}(\gamma \delta^{-1}(h) k) = T_{\gamma \delta} \pi(\delta^{-1}(h) k)$ we get

\[ \tilde{\pi}(\gamma, h) \tilde{\pi}(\delta, k) = \sigma(\gamma, \delta) \tilde{\pi}((\gamma, h)(\delta, k)). \]

Therefore $\tilde{\pi}$ is a projective representation of $G$ with cocycle $\sigma$.

We recall now the definition of compact induction, $c$-Ind, as we will use it: Let $L$ be a an open subgroup of $H$, compact modulo the centre of $H$, and let $(\rho, W)$ be a smooth representation of $L$. Let $V$ denote the space of compactly supported modulo the centre of $H$ functions $f : H \to W$ with the property $f(lh) = \rho(l)f(h)$, $l \in L, h \in H$. The group acts on this space by right translation of functions; the implied representation is smooth. We will assume now that

$(\pi, V) = c - \text{Ind}^H_L \rho$, where $L$ is an open, compact modulo the centre, subgroup of $H$ and $\rho$ is a one dimensional representation of $L$.

We assume also that $\rho^\gamma = \rho$ on $L^\gamma \cap L$, where $L^\gamma = \gamma(L)$ and $\rho^\gamma(y) = \rho(\gamma^{-1}(y))$ with $y \in L^\gamma$. We can define, similarly,

\[ (\pi^\gamma, V^\gamma) = c - \text{Ind}^H_{L^\gamma} \rho^\gamma. \]

Let $S_\gamma$ be a non zero intertwining operator from $(\pi, V)$ to $(\pi^\gamma, V^\gamma)$. So $S_\gamma$ is an isomorphism between $\pi$ and $\pi^\gamma$ when $\pi$ (and then $\pi^\gamma$) is irreducible. Then $S_\gamma \pi(x) = \pi^\gamma(x) S_\gamma$. 
We define now $I_\gamma : V_{\gamma^{-1}} \rightarrow V$ by $(I_\gamma(f))(x) = f(\gamma^{-1}(x))$. The operator $I_\gamma$ is well defined and intertwining, in fact, $I_\gamma(f(lx)) = \rho(l)f(x)$ and $I_\gamma \pi_{\gamma^{-1}}(x) = \pi(\gamma(x))I_\gamma$. On the other hand, we have that $I_\gamma S_\gamma : V \rightarrow V$ is an intertwining operator since $I_\gamma S_\gamma \pi(x) = \pi(\gamma(x))I_\gamma S_\gamma$. Let us define $T_\gamma = I_\gamma S_\gamma$. We want to compute the cocycle $\sigma$. In order to do this we look first at $I_\gamma$ on $V_{\delta}$. Since $\gamma^{-1}(h) \in \delta(L)$ implies that $h \in \gamma \delta(L)$, we have $(I_\gamma f)(hx) = f(\gamma^{-1}(h)\gamma^{-1}(x))$.

We can define $I_\sigma : V_{\gamma^{-1}\delta} \rightarrow V_{\delta}$ by $(I_\sigma f)(x) = f(\gamma^{-1}\delta x)$, and $S_{\delta,\gamma} : V_{\gamma^{-1}\delta} \rightarrow V_{\gamma^{-1}\delta}$ by $S_{\delta,\gamma} = I_{\gamma^{-1}\delta}^{-1} S_{\delta} I_{\gamma^{-1}}$ a computation shows that $S_{\delta,\gamma}$ is an intertwining map.

Since the operators $S_{\delta,\gamma} \circ S_\gamma : V \rightarrow V_{\gamma^{-1}\delta^{-1}}$ and $S_{\delta,\gamma} : V \rightarrow V_{\gamma^{-1}\delta^{-1}}$ are both intertwining, the irreducibility of $V$ implies that they differ on a scalar i.e. $S_{\delta,\gamma} \circ S_\gamma = kS_{\delta,\gamma}$.

**Lemma 2.2.** The intertwining operators defined above satisfy the equation $I_\delta \circ I_{\gamma^{-1}\delta^{-1}} = I_{\delta\gamma}$.  
**Proof.** Straightforward.

We finally show that $k = \sigma(\delta, \gamma)$: Since $S_{\delta,\gamma} \circ S_\gamma = kS_{\delta,\gamma}$ we have $I_{\gamma^{-1}\delta}^{-1} S_{\delta} I_{\gamma^{-1}} S_\gamma = k S_{\delta,\gamma}$. So $S_{\delta} I_{\gamma^{-1}} S_\gamma = k I_{\gamma^{-1}\delta} S_{\delta,\gamma}$ and then $I_\delta S_{\delta} I_{\gamma^{-1}} S_\gamma = k I_\delta I_{\gamma^{-1}} S_{\delta,\gamma}$. Using Lemma 2.2 we get $I_\delta S_{\delta} I_{\gamma^{-1}} S_\gamma = k I_{\delta,\gamma} S_{\delta,\gamma}$ i.e. $T_{\delta} T_\gamma = k T_{\delta,\gamma}$.

3. Heisenberg Construction

Given a finite vector space $W$ we can define $H = F \oplus W$ which has a structure of group with respect to

$$(a, w) \cdot (a', w') = (a + a' + B(w, w'), w + w')$$

where $B : W \times W \rightarrow F$ is a non-degenerate alternating form.

If $M$ is any subgroup of $W$ we write $M = F \oplus M$, which is a subgroup of $H$.

**Definition 3.1.** Let $M$ be an any subset of $W$. We define $M^* = \{w \in W \mid B(m, w) \in O_F \forall m \in M\}$ and $M^\perp = \{w \in W \mid B(m, w) = 0 \forall m \in M\}$.

**Observation 3.2.**

a) If $M$ is a $F$--subspace of $W$, then $M^* = M^\perp$. In fact, the inclusion $M^\perp \subset M^*$ is obvious. On the other hand, since $\alpha B(m, w) = B(\alpha m, w)$ we have that $w \in M^*$ implies that $\alpha B(m, w) \in O_F \forall m \in M \forall \alpha \in F$, so $B(m, w) = 0$.

b) Another fact that we will use later, is the following

$$[(a, w), (a', w')] = (2B(w, w'), 0).$$

Let $\mathcal{L}$ be a maximal isotropic lattice i.e. $\mathcal{L}$ is compact and open and $\mathcal{L}^* = \mathcal{L}$. Set $\tilde{\mathcal{L}} = F \oplus \mathcal{L}$ and let $\psi$ be a character of $F$ of conductor $O_F$. Define $\psi_{\tilde{\mathcal{L}}}$ on $\tilde{\mathcal{L}}$ by $\psi_{\tilde{\mathcal{L}}}(a, l) = \psi(a)$ for $a \in F$.

**Proposition 3.3.** With the above notation and assuming that $2 \in O_F^*$ we have:

a) $\psi_{\tilde{\mathcal{L}}}$ is a character of $\tilde{\mathcal{L}}$.

b) If we define $\text{Int}_H(\psi_{\tilde{\mathcal{L}}}) = \{h \in H \mid \text{Hom}_{\tilde{\mathcal{L}} \oplus \tilde{\mathcal{L}}} (\psi_{\tilde{\mathcal{L}}}, \psi_{\tilde{\mathcal{L}}}^h) \neq 0\}$, where $\tilde{\mathcal{L}}^h = h\tilde{\mathcal{L}}h^{-1}$
and $\psi^h(x) = \psi_\mathcal{E}(h^{-1}xh)$ for any $x \in \bar{\mathcal{E}}^h$, then $\text{Int}_H(\psi_\mathcal{E})$ is equal to $\bar{\mathcal{E}}$.

**Proof.**

a) $\psi_\mathcal{E}(a, w)(a', w') = \psi_\mathcal{E}(a + a' + B(w, w'), w + w')$, since $\mathcal{E}$ is a maximal isotropic lattice, $B(w, w') \in O_F$. Then $\psi_\mathcal{E}(a, w)(a', w') = \psi(a)\psi(a') = \psi_\mathcal{E}(a, w)\psi_\mathcal{E}(a', w')$.

b) If $(a, w) \in H$ Since $(-a, -w)(\alpha, y)(a, w) = (\alpha + 2B(y, w), y)$ and $\bar{\mathcal{E}} \triangleleft H$, we have $\psi^{(a, w)}_\mathcal{E} = \psi_\mathcal{E}$ on $\mathcal{E} \cap (-a, -w)\mathcal{E}(a, w) = \bar{\mathcal{E}}$ if and only if $2B(y, w) \in O_F$ for all $y \in \mathcal{E}$ and only if $B(y, w) \in O_F$ for all $y \in \mathcal{E}$ (given that $2 \in O_F$) and this is the case if and only if $w \in \mathcal{E}$.

Now let $\Pi_\mathcal{E} = c - \text{Ind}^H_\mathcal{E} \psi_\mathcal{E}$ be the compact induction of the character $\psi_\mathcal{E}$ from $\bar{\mathcal{E}}$ to $H$ as defined in Section 2.

**Proposition 3.4.** The representation $\Pi_\mathcal{E}$ defined above is an irreducible admissible supercuspidal representation of $H$.

**Proof.** The representation $\Pi_\mathcal{E}$ is the Heisenberg representation realized in the lattice model (see [5], Chapter 2). Stone-von Neumann theorem implies that $\Pi_\mathcal{E}$ is a smooth irreducible (thus admissible) representation. Then, using theorem 1 of [2], we get that it is supercuspidal.

Now let $\Gamma$ be the subgroup of $\text{Aut}(H)$ of all automorphism $\gamma : H \rightarrow H$ such that $\gamma_F = id_F$ and $\gamma_W$ is a symplectic linear automorphism. The subgroup $\Gamma$ acts transitively over the set $\Theta$ of all maximal isotropic lattices in $W$, by $\mathcal{E}^\gamma = \gamma(\mathcal{E})$ ($\gamma \in \Gamma$ and $\mathcal{E} \in \Theta$). Furthermore $\psi^\gamma_\mathcal{E} = \psi_\mathcal{E}$ on $\mathcal{E} \cap \mathcal{E}$ where $\psi^\gamma_\mathcal{E}(y) = \psi_\mathcal{E}(\gamma^{-1}(y))$, $\forall y \in \mathcal{E}^\gamma$.

On the other hand, by Proposition 3.4, $(\Pi_\mathcal{E}, V_\mathcal{E}) = c - \text{Ind}^H_\mathcal{E} \psi_\mathcal{E}$ is an irreducible admissible supercuspidal representation of $H$, where $V_\mathcal{E} = \{ f : H \rightarrow \mathbb{C} | f(lx) = \psi_\mathcal{E}(l)f(x), \forall l \in \mathcal{E}, \forall x \in H, f \text{ compactly supported modulo the centre of } H \}$. So, we can define $(\Pi_\mathcal{E}^\gamma, V_\mathcal{E}^\gamma) = c - \text{Ind}^H_\mathcal{E} \psi_\mathcal{E}^\gamma$, where $V_\mathcal{E}^\gamma = \{ f : H \rightarrow \mathbb{C} | f(lx) = \psi_\mathcal{E}^\gamma(l)f(x), \forall l \in \mathcal{E}^\gamma \}$ and now the general set-up of Section 2 applies.

Define the function $\tau_\gamma : H \rightarrow \mathbb{C}$ by

$$
\tau_\gamma(xy) = \begin{cases} 
\psi_\mathcal{E}(x)\psi_\mathcal{E}(y) & \text{if } x \in \mathcal{E}, y \in \mathcal{E} \\
0 & \text{otherwise}
\end{cases}
$$

Note that $\tau_\gamma$ is well defined since $\psi_\mathcal{E} = \psi_\mathcal{E}$ on $\mathcal{E} \cap \mathcal{E}$. For any $f$ in the space of $\Pi_\mathcal{E}$ we can define $\Upsilon_\gamma(f) : H \rightarrow \mathbb{C}$ by

$$
\Upsilon_\gamma(f)(x) = \int_{H/F} \tau_\gamma(y)f(y^{-1}x)dy
$$

for an appropriate Haar measure on $W = H/F$. We can observe that $\Upsilon_\gamma : V_\mathcal{E} \rightarrow V_{\mathcal{E}^{-1}}$ is a non-zero intertwining operator and since $\Pi_\mathcal{E}$ is irreducible (and also $\Pi_{\mathcal{E}^{-1}}$), we have that $\Upsilon_\gamma$ is an isomorphism.

We define now $I_\gamma : V_{\mathcal{E}^{-1}} \rightarrow V_\mathcal{E}$ by $(I_\gamma f)(x) = f(\gamma^{-1}(x))$ and so we have, as in section 2, that $T_\gamma = I_\gamma \Upsilon_\gamma$ is an intertwining of $V_\mathcal{E}$ which verify

$$
T_\delta \circ T_\gamma = \sigma(\delta, \gamma)T_\delta.
$$
4. Lagrangians

Let $S$ be a left $A$-module whose $F$-dimension is $n$. We note that $S$ is a right $A$-module with $sa = a^*s$, $a \in A$, $s \in S$.

Let $b : S \times S \rightarrow F$ be a non degenerate bilinear symmetric form such that

$$b(x_1a, x_2) = b(x_1, ax_2) \quad (a \in A; x_1, x_2 \in S).$$

We set now $W = S \oplus S$ and define $B : W \times W \rightarrow F$ by $B(x, y) = b(x_1, y_2) - b(y_1, x_2)$ for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $W$. Observe that $B$ is a non degenerate alternating form and we can define $M^\perp = \{w \in W \mid B(w, m) = 0, \forall m \in M\}$ for any $O_F$-submodule $M$ of $W$. The following properties are straightforward.

1. If either $M$ is an $F$-subspace or if $M$ is a compact open $O_F$-submodule of $W$ (an $O_F$-lattice in $W$) then $(M^\perp)^\perp = M$.
2. If $M, N$ are any $O_F$-submodules which satisfy $(M^\perp)^\perp = M$ and $(N^\perp)^\perp = N$ then $(M \cap N)^\perp = M^\perp + N^\perp$.

We call an $O_F$-submodule $M$ of $W$ isotropic if $M$ is an $O_F$-submodule of $M^\perp$. We say that $M$ is maximal isotropic if $M = M^\perp$.

Fixing an additive (continuous) character $\psi$ of $F$ of conductor $O_F$, we can define the function $\chi : W \times W \rightarrow \mathbf{T}$, where $\mathbf{T}$ is the group of complex numbers of module one, by

$$\chi(x, y) = (\psi \circ B)(x, y) \quad ((x, y) \in W \times W).$$

which is a symplectic bicharacter.

**Definition 4.1.** Let $M$ be a subset of $W$. The orthogonal component $M^*$ of $M$ is the set of $y \in W$ such that $\chi(x, y) = 1$, for every $x \in M$.

**Observation 4.2.** In the case where $M$ is a $F$-subspace of $W$ we have that $M^*$ is also a $F$-subspace of $W$ and $M^\perp = M^*$. 

**Definition 4.3.** Let $L$ be a $F$-subspace of $W$ such that $L^\perp = L$. $L$ is called a Lagrangian subspace of $W$.

**Observation 4.4.** If $M$ is an $F$-subspace of $W$ then $M$ is maximal isotropic if and only if $M$ is Lagrangian.

**Lemma 4.5.** Let $W$ and $\chi$ be as above. Let $L$ and $L'$ be two Lagrangian subspaces of $W$. Then there exists a symplectic basis $\{w_1, w_2, \ldots, w_n, w'_1, w'_2, \ldots, w'_n\}$ of $W$, i.e.

1. $\chi(w_j, w'_j) \neq 1$, $j = 1, \ldots, n$
2. $\chi(w_i, w_j) = \chi(w'_i, w'_j) = 1$ for every $i, j$.
3. $\chi(w_i, w'_j) = 1$ for every $i \neq j$.
such that:
\[
L = Fw_1 + Fw_2 + \cdots + Fw_k + Fw_{k+1} + \cdots + Fw_n
\]
\[
L' = Fw_1' + Fw_2' + \cdots + Fw_k' + Fw_{k+1} + \cdots + Fw_n
\]

**Proof.** See Lemma 1.4.6. in [4]. □

**Corollary 4.6.** Given a Lagrangian \( L \), there exists a Lagrangian \( L' \) such that \( W = L \oplus L' \).

**Proof.** If \( L = \langle w_1, w_2, \ldots, w_n \rangle \), then \( L \) is a proper subspace of \( \langle w_2, \ldots, w_n \rangle \). We consider an element \( v_1 \in \langle w_2, \ldots, w_n \rangle - L \). Then \( \chi(w_1, v_1) \neq 1 \). Now we can pick an element \( v_2 \in \langle w_1, w_3, w_4, \ldots, w_n, v_1 \rangle - \langle w_1, w_2, w_3, \ldots, w_n, v_1 \rangle \), and so \( \chi(w_2, v_2) \neq 1 \). By induction we have \( \{w_1, v_1\}, \{w_2, v_2\}, \ldots, \{w_n, v_n\} \) such that \( \chi(w_i, v_i) \neq 1 \), \( i = 1, \ldots, n \); \( \chi(w_i, w_j) = \chi(v_i, v_j) = 1 \) for every \( i, j \) and \( \chi(w_i, v_j) = 1 \) for every \( i \neq j \). Hence \( L' = \langle v_1, v_2, \ldots, v_n \rangle \) is such that \( W = L \oplus L' \). □

**Corollary 4.7.** There exists a maximal isotropic \( O_F \)-lattice \( \mathfrak{L} \) in \( W \).

Let \( L \) be a Lagrangian in \( W \) and define \( \psi_L \) on \( \tilde{L} = F \oplus L \) as above. Let \( \Pi_L = c - \text{Ind}_L^H \psi_L \) and consider the group \( H = F \oplus W \). Let \( \tilde{\mathfrak{L}} = F \oplus \mathfrak{L} \), \( \mathfrak{L} \) a maximal isotropic \( O_F \)-lattice in \( W \).

Now we can define the function \( \rho : H \to C \) by
\[
\rho(z) = \begin{cases} 
\psi_L(x)\psi_L(y) & \text{if } z = x \cdot y, \ x \in \tilde{L}, \ y \in \tilde{\mathfrak{L}} \\
0 & \text{if } z \notin \tilde{L} \oplus (L + \mathfrak{L})
\end{cases}
\]

Note that \( \rho \) is well defined since \( \psi_L = \psi_L \) on \( \tilde{L} \cap \tilde{\mathfrak{L}} \) and \( \tilde{L} \cap \tilde{\mathfrak{L}} = F \oplus (L \cap \mathfrak{L}) \).

For any \( f \) in the space of \( \Pi_L \) we can define \( S(f) : H \to C \) by
\[
S(f)(x) = \int_{H/F} \rho(y)f(y^{-1}x)dy.
\]

Given an \( O_F \)-lattice \( \mathfrak{M} \) submodule of \( \mathfrak{L} \), we define the function
\[
\rho_{\mathfrak{M}}(z) = \begin{cases} 
\psi_L(x)\psi_L(y) & \text{if } z = x \cdot y, \ x \in \tilde{L}, \ y \in \tilde{\mathfrak{M}} \\
0 & \text{if } z \notin \tilde{L} \oplus \mathfrak{M} \oplus (L + \mathfrak{M})
\end{cases}
\]

**Proposition 4.8.** The map \( S \) defined above is an \( H \)-isomorphism from \( \Pi_L \) to \( \Pi_L \)

**Proof.** Let \( f_0 \) be the function, in the space of \( \Pi_L \), defined by
\[
f_0(z) = \begin{cases} 
\psi_L(z) & \text{if } z \in \tilde{\mathfrak{L}} \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
f_M(z) = \begin{cases} 
\psi_M(z) & \text{if } z \in \tilde{M} \\
0 & \text{otherwise}
\end{cases}
\]

A computation shows \( S(f_0) = \rho \) and \( S(f_M) = \rho_M \).

Since \( S \) is different from 0 and \( \Pi_L \) is irreducible, \( S \) is injective.

We will prove now that \( S \) is onto. To this end we prove that the space of \( \Pi_L \) is equal to \( \langle \{ \rho_M \mid M \subset L \} \rangle \). First, \( S(f_M) = \rho_M \) so \( \langle \{ \rho_M \mid M \subset L \} \rangle \subset \Pi_L \).

On the other hand any \( f \) in \( \Pi_L \) has support compact modulo \( \tilde{L} \) and it is locally constant. From this, it can be seen that any function \( f \) is a linear combination of \( \rho_M \) for different lattices \( M \subset L \). Hence we can conclude that \( S \) is an isomorphism.

Define now \( T : \Pi_L \longrightarrow \Pi_L \) by
\[
T(f)(x) = \int_{H/F} \theta(y)f(y^{-1}x)dy
\]
where \( \theta \) is given by
\[
\theta(z) = \begin{cases} 
\psi_L(x)\psi_L(y) & \text{if } z = x \cdot y, \ x \in \mathcal{L}, y \in L \\
0 & \text{if } z \notin \mathcal{L}L
\end{cases}
\]
We have that \( T \neq 0 \) and by Schur’s Lemma [1] [3], \( TS = cI \), so \( TS(f_0) = cf_0 \) which implies \( c = 1 \), and finally
\[
TS = I_{\Pi_L}
\]

5. Connections over \( SL_*(2,A) \).

The group \( G = SL_*(2,A) \) acts naturally by matrix multiplication on \( W \) by fixing the bicharacter \( \chi \),
\[
\chi(gx,gy) = \chi(x,y) \quad (x,y \in W)
\]

We define a complex \( G \)-bundle space \( \mathfrak{F} = (\mathcal{E}, p, \Gamma, \tau) \) by:

1. \( \Gamma = \{ L \mid L \text{ a Lagrangian of } W \} \)

2. Fix a Haar measure \( dw \) on \( W \) and \( dw_L \) on a Lagrangian \( L \) such that \( d\overline{w_L} \) is the unique Haar measure on \( W/L \) which verifies that \( dw = d\overline{w_L}dw_L \).

For each Lagrangian \( L \) we consider the set \( \mathcal{E}_L \) of all functions \( f : W \longrightarrow \mathbb{C} \) which are locally constant, compactly supported modulo \( L \), and such that \( f(w + l) = \chi(w,l)f(w) \) for every \( w \in W \) and \( l \in L \).

We set
\[
\mathcal{E} = \bigcup_{L \in b} \mathcal{E}_L
\]
and we define an inner product on each \( \mathcal{E}_L \) by
\[
\langle f, h \rangle = \int_{W/L} f(w)\overline{h(w)}dw_L \quad (f, h \in \mathcal{E}_L)
\]
3. Let \( p : \mathcal{E} \longrightarrow \Gamma \) be the canonical projection which sends each \( f \) of \( \mathcal{E}_L \) to \( L \).

4. The group \( G \) acts on \( \mathcal{E} \) and \( \Gamma \) by

\[
[\tau_g(f)](w) = f(g^{-1}w) \quad (f \in \mathcal{E}, \ g \in G, \ w \in W)
\]

and by

\[
\tau_g(L) = gL \quad (L \in \mathfrak{b}, \ g \in G)
\]

respectively.

**Lemma 5.1.** Let \( L \) be a Lagrangian subspace of \( W \). Let \( M \) be an \( O_F \)-lattice of \( W \). We set

\[
g_M(w) = \begin{cases} 
\chi(x,c) & \text{if } w = x + c \in L + M \\
0 & \text{otherwise.}
\end{cases}
\]

Then, the set \( \{g_M \mid M \text{ be an } O_F \text{-lattice of } L\} \) span \( \mathcal{E}_L \) as a \( \mathbb{C} \)-vector space.

**Proof.** For each \( f \) in \( \mathcal{E}_L \) we can pick an \( O_F \)-lattice \( M \) such that \( \text{Supp}(f) = L + M \). We use that \( f \) is locally constant and \( M \) is compact, to write \( f \) as linear combination of \( g_M \)'s as above.

Let \( L \) and \( L' \) be Lagrangians included in a fixed maximal \( O_F \)-lattice \( \mathcal{L} \) in \( W \). As we have seen, there are two isomorphisms, namely \( S_L : \Pi_L \longrightarrow \Pi_L \) and \( S_{L'} : \Pi_L \longrightarrow \Pi_L' \) with \( T_L : \Pi_L \longrightarrow \Pi_L \) and \( T_{L'} : \Pi_L' \longrightarrow \Pi_L \) as the respective inverses.

We now define isomorphisms \( \tilde{\gamma}_{L',L} : \Pi_L \longrightarrow \Pi_{L'} \), by

\[
\tilde{\gamma}_{L',L} = S_{L'} \circ T_L
\]

Let \( \Lambda^L : \Pi_L \longrightarrow \mathcal{E}_L \) be defined by \( \Lambda^L(f)(w) = f(0, w) \), for \( f \in \Pi_L \) and \( w \in W \), and let, \( \Omega^L : \mathcal{E}_L \longrightarrow \Pi_L \) be defined by \( \Omega^L(f)(a, w) = \psi(a)f(w) \), for \( f \in \mathcal{E}_L \) and \((a, w) \in \tilde{L} \). A computation shows that \( \Lambda^L \) and \( \Omega^L \) are inverse to each other and both are intertwining operators.

We can define now isomorphisms (which we will call connections) \( \gamma_{L,L'} : \mathcal{E}_L \longrightarrow \mathcal{E}_{L'} \) by \( \gamma_{L,L'} = \Lambda^L' \circ \tilde{\gamma}_{L,L'} \circ \Omega^L \).

Then the diagram

\[
\begin{array}{ccc}
\Pi_L & \longrightarrow & \Pi_{L'} \\
\Omega^L \uparrow & & \downarrow \Lambda^L' \\
\mathcal{E}_L & \longrightarrow & \mathcal{E}_{L'}
\end{array}
\]

is commutative.

We obtain

**Theorem 5.2.** The set \( \Gamma = \{\gamma_{L',L} \mid L', L \in \mathfrak{b}\} \) is a family of \( G \)-equivariant connections over the fiber bundle \( \mathfrak{F} \) which verifies, for \( L, L', L'' \in \mathfrak{b}; \ f, f' \in \mathcal{E}_L; \ h \in \mathcal{E}_{L'}; \ g \in G \) the following properties:

1. \( \gamma_{L,L'} \circ \gamma_{L',L} = \gamma_{L,L} = id_{\mathcal{E}_L} \)
1. $\langle \gamma_{L',L}(f), h \rangle = \langle f, \gamma_{L,L}(h) \rangle$

2. $\langle \gamma_{L',L}(f), \gamma_{L',L}(f') \rangle = \langle f, f' \rangle$

3. $\gamma_{L,L''} \circ \gamma_{L',L'} \circ \gamma_{L',L} = S_W(L; L', L'') \text{id}_E$

4. where $S_W(L; L', L'')$ is a constant.

5. $\tau_g \circ \gamma_{L',L} = \gamma_{gL',gL} \circ \tau_g$

Note that $S_W(L; L', L'')$ is the analogous of the Maslov index in [4] and this theorem is comparable with theorem 1.4 in [6].

References


