Lengths of Involutions in Coxeter Groups

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Abstract. Let \( t \) be an involution in a Coxeter group \( W \). We determine the minimal and maximal (in the case of finite \( W \)) length of an involution in the conjugacy class of \( t \).

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Let \( W \) be a finitely generated Coxeter group whose distinguished set – the set of fundamental reflections – is \( R \). The length \( l(w) \) of a non-trivial element \( w \) in \( W \) is defined to be

\[
l(w) = \min\{l \in \mathbb{N} : w = r_1r_2 \cdots r_l \text{ some } r_i \in R\}
\]

and \( l(1) = 0 \). Suppose \( t \) is an involution in \( W \), and let \( C = tW \) be the conjugacy class of \( t \) in \( W \). The aim of this short paper is to determine the minimal and maximal (in which case \( W \) is assumed finite) length of an involution in \( C \).

Associated to any Coxeter group \( W \) is the root system \( \Phi \), which is the disjoint union of its positive and negative roots (denoted \( \Phi^+ \) and \( \Phi^- \) respectively). The fundamental reflections \( r \in R \) are in one-to-one correspondence with the fundamental roots \( \alpha_r, r \in R \) and \( W \) acts faithfully on \( \Phi \) (see [1]). For \( w \in W \), define \( N(w) := \{\alpha \in \Phi^+ : w \cdot \alpha \in \Phi^-\} \), \( I(w) := \{\alpha \in \Phi^+ : w \cdot \alpha = -\alpha\} \) and \( \text{Fix}(w) := \{\alpha \in \Phi^+ : w \cdot \alpha = \alpha\} \). It is well known that for each \( w \in W \), \( l(w) = |N(w)| \). For \( J \subseteq R \), write \( W_J \) for the (Coxeter) group generated by \( J \), \( \Phi_J \) for its root system and, when it is finite, \( w_J \) for the unique longest element of \( W_J \). Our main result is given in

Theorem 1.1. Suppose \( t \) is an involution in \( W \), and put \( C = tW \). We have

\begin{enumerate}
  \item \( \min_{s \in C}\{l(s)\} = |I(t)| \) and if \( x \) is of minimal length in \( C \), then \( x = w_J \) for some \( J \subseteq R \).
  \item If \( W \) is finite, then \( \max_{s \in C}\{l(s)\} = |\Phi^+| - |\text{Fix}(t)| \) and for \( y \) of maximal length in \( C \), \( y = w_Kw_R \) for some \( K \subseteq R \).
\end{enumerate}

Put another way, Theorem 1.1 is saying that the maximum and minimum length in a conjugacy class of involutions may be obtained by examining the action on \( \Phi \).
of any one involution in that class. We remark that part (i) appears as Theorem A (a) in [3]. We include a (shorter, and different) proof here to emphasise the similarity between parts (i) and (ii).

**Proof.** Let $t$ be an involution and $C = t^W$. Note that for any $t' \in C$, $|I(t')| = |I(t)|$ and $|\text{Fix}(t')| = |\text{Fix}(t)|$, because $t \cdot \alpha = \pm \alpha$ if and only if $t^g \cdot (g \cdot \alpha) = \pm (g \cdot \alpha)$, for each $g \in W$. It is clear from this that the length of any involution in $C$ is at least $|I(t)|$ and at most $|\Phi^+| - |\text{Fix}(t)|$. Let $r \in R$ with $\alpha_r \notin N(t)$, and suppose $\alpha_r \notin \text{Fix}(t)$. It is well known that for any $w \in W$, $r \in R$, $l(wr) > l(w)$ if and only if $w \cdot \alpha_r \in \Phi^+$. We have $t \cdot \alpha_r \in \Phi^+ \setminus \{\alpha_r\}$, so $rt \cdot \alpha_r \in \Phi^+$. Therefore $l(rtr) > l(rt)$. Now $rt = (tr)^{-1}$, hence $l(rt) = l(tr) > l(t)$, since $\alpha_r \notin N(t)$. Thus $l(rtr) > l(t)$. Suppose now that $\alpha_r \in N(t)$ with $\alpha_r \notin I(t)$. We have $l(rtr) < l(rt)$ because $rt \cdot \alpha_r \in \Phi^-$, and $l(rt) = l(tr) < l(r)$ because $\alpha_r \in N(t)$. Thus $l(rtr) < l(t)$.

We have shown that if $\alpha_r \notin N(t)$, then either $l(rtr) > l(t)$ or $\alpha_r \in \text{Fix}(t)$, and that if $\alpha_r \in N(t)$, then either $l(rtr) < l(t)$ or $\alpha_r \in I(t)$. Thus for each $x$ of minimal length in $C$, there exists $J \subseteq R$ with $\alpha_r \in I(x)$ for each $r \in J$ and $\alpha_r \notin N(x)$ when $r \notin J$. Let $r \in J$. Then $w_j x \cdot \alpha_r = -w_j \cdot \alpha_r \in \Phi^+$. If $r \notin J$ then $w_j x \cdot \alpha_r \in \Phi^+$ unless $x \cdot \alpha_r \in \Phi^+$. But this would imply that $x^2 \cdot \alpha_r = -x \cdot \alpha_r \neq \alpha_r$, which is impossible. Thus $N(w_j x) = \emptyset$ and hence $x = w_j$. Now $N(x) = \Phi^+_J = I(x)$ and so $x$ has length $|I(t)|$ in $C$, which is minimal.

Similarly, when $W$ is finite, for $y$ of maximal length in $C$ there exists $K \subseteq R$ with $\alpha_r \in \text{Fix}(y)$ whenever $r \in K$, and $\alpha_r \in N(y)$ for $r \notin K$. We claim that $\text{Fix}(y) = \Phi^+_K$. Certainly $\Phi^+_K \subseteq \text{Fix}(y)$. For the reverse inclusion, let $\alpha = \sum_{r \in K} \lambda_r \alpha_r \in \text{Fix}(y)$ (where each $\lambda_r \geq 0$). Now $y \cdot \alpha_r \in \Phi^-$ for all $r \in R \setminus K$, so $\sum_{r \in R \setminus K} \lambda_r y \cdot \alpha_r$ is a negative linear combination of roots, say $-\sum_{r \in K} \mu_r \alpha_r$ for some $\mu_r \geq 0$. We have $\sum_{r \in R \setminus K} \lambda_r \alpha_r = \alpha = y \cdot \alpha = \sum_{r \in K} (\lambda_r - \mu_r) \alpha_r - \sum_{r \in R \setminus K} \mu_r \alpha_r$. For $r \in R \setminus K$ then, we see that $\lambda_r = -\mu_r$. Hence $\lambda_r = \mu_r = 0$. Therefore $\alpha \in \Phi^+_K$ and so $\text{Fix}(y) \subseteq \Phi^+_K$.

Now for $r \in K$, $w_K y \cdot \alpha_r = w_K \cdot \alpha_r \in \Phi^-$. If $r \notin K$, $w_K y \cdot \alpha_r \in \Phi^+$ only when $y \cdot \alpha_r \in \Phi^+_K$, which is impossible. Consequently $N(w_K y) = \Phi^+$, that is $y = w_K w_R$ and $l(y) = |N(y)| = |\Phi^+| - |\Phi^+_K| = |\Phi^+| - |\text{Fix}(y)|$ and this is the maximum possible length of an involution in $C$.

We remark that it is necessary, as Proposition 1.3 shows, to assume, when $W$ is irreducible, that $W$ is finite in order for $\max_{s \in C} \{l(s)\}$ to be defined. We require the following lemma, which follows from the fact that the geometric representation of $W$ is irreducible and faithful (see [1]).

**Lemma 1.2.** ([3], Lemma 2.3) Let $W$ be an irreducible Coxeter group and let $\alpha \in \Phi$. Then $W$ acts faithfully on the orbit $W \cdot \alpha$.

**Proposition 1.3.** Suppose $W$ is an infinite irreducible Coxeter group. Then each conjugacy class of involutions in $W$ contains elements of arbitrarily large length.

**Proof.** Let $t$ be an involution of $W$. Then, by Theorem 1.1, $I(t)$ is non-empty, so there exists $\alpha \in \Phi^+$ with $t \cdot \alpha = -\alpha$. Let $\beta \in W \cdot \alpha$. Then $\beta = w \cdot \alpha$ for
some $w \in W$. Now $t^w \cdot \beta = wtw^{-1} \cdot (w \cdot \alpha) = -\beta$, whence $\beta \in N(t^w)$. Thus $W \cdot \alpha \subseteq \cup_{w \in W} N(t^w)$. Each element $t^w$ has finite length, but $W \cdot \alpha$ is infinite, by Lemma 1.2, hence the conjugacy class of $t$ must be infinite. Consequently, since there can only be finitely many elements of a given length in $W$, the conjugacy class of $t$ must contain elements of arbitrarily large length.

References