A monogenic Hasse-Arf theorem

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RÉSUMÉ. On étend le théorème de Hasse–Arf de la classe des extensions résiduellement séparables des anneaux de valuation discrète complets à la classe des extensions monogènes.

Abstract. I extend the Hasse–Arf theorem from residually separable extensions of complete discrete valuation rings to monogenic extensions.

Let \( B/A \) be a finite extension of henselian discrete valuation rings which is generically Galois with group \( G \), that is, for which the corresponding extension of fraction fields is Galois with group \( G \). For \( \sigma \in G - \{1\} \), let \( I_B(\sigma) \) be the ideal of \( B \) generated by \((\sigma - 1)B\) and let \( i_B(\sigma) \) be the length of the \( B \)-module \( B/I_B(\sigma) \).

For any finite dimensional complex representation \( \rho : G \to \text{Aut}_C(V) \), we define the naive Artin conductor exactly as we do when \( B/A \) is residually separable, i.e., when the extension of residue fields is separable:

\[
\text{ar}_n(\rho) = e_{B/A}^{-1} \sum_{\sigma \neq 1} [\dim(V) - \text{trace}(\rho(\sigma))]i_B(\sigma).
\]

By looking at real parts, it is immediate that this is a non-negative rational number, and when \( B/A \) is residually separable, the Hasse-Arf theorem [3, VI §2] tells us that it is also an integer.

In [4], De Smit shows that most of the classical ramification-theoretic properties of residually separable extensions \( B/A \) hold in the slightly more general, “monogenic” case where we require only that \( B \) is generated as an \( A \)-algebra by one element. The purpose of this note is to show that the Hasse–Arf theorem also holds in this context.

Partial results in this direction were obtained by Spriano [5]. A proof of the Hasse-Arf theorem in equal characteristic that is strong enough to cover monogenic extensions was outlined at the 1999 Luminy conference on ramification theory. It was based on a technical analysis of a refinement [2, 3.2.2] of Kato’s refined Swan conductor [1], but since then, an elementary reduction to the classical Hasse-Arf theorem has been found.

The contents of this paper are contained in my dissertation (U.C. Berkeley, 2000), which was written under the direction of Hendrik Lenstra.
Proposition 1. Let $B/A$ be a finite generically separable extension of henselian discrete valuation rings. Then the following are equivalent.

(i) There exists an $x \in B$ such that $B = A[x]$.

(ii) The second exterior power $\Omega^2_{B/A}$ of the module of relative Kähler differentials is zero.

(iii) There is a henselian discrete valuation ring $A'$ that is finite over the maximal unramified subextension $A^{ur}$ of $B/A$ such that $e_{A'/A^{ur}} = 1$ and $B'/A'$ is a residually separable extension of discrete valuation rings, where $B' = A' \otimes_{A^{ur}} B$.

Proof. De Smit [4, 4.2] shows that (i) follows from (ii). For any $A'$ as in (iii), we have $B' \otimes_B \Omega^2_{B/A} \cong B' \otimes_B \Omega^2_{B/A^{ur}} \cong \Omega^2_{B'/A'} = 0$, so (iii) implies (ii). Now we show (i) implies (iii).

Assume, as we may, that $A = A^{ur}$, and let $l/k$ denote the residue extension of $B/A$. Take some $x \in B$ such that $B = A[x]$ and let $\bar{x}$ denote the image of $x$ in $l$. Let $g(X) \in A[X]$ be a monic lift of the minimal polynomial $X^n - a$ of $\bar{x}$ over $k$. Since the maximal ideal of $B$ is generated by that of $A$ and $g(x)$, we may assume that $g(x)$ generates the maximal ideal of $B$. Then modulo the maximal ideal of $B$, we have $g(X + x) \equiv X^n + x^n - a \equiv X^n$, so $g(X + x)$ is an Eisenstein polynomial with coefficients in $B$. Now let $A'$ be the discrete valuation ring $A[X]/(g(X))$. Then

$$B' = A' \otimes_A B \cong B[X]/(g(X)) \cong B[X]/(g(X + x))$$

is a discrete valuation ring which has the same residue field as $B$ and, hence, $A'$.

\[\square\]

Proposition 2. Let $B/A$ be a finite extension of henselian discrete valuation rings that is generically Galois with group $G$, and let $\rho : G \to \text{Aut}_{C(V)}$ be a finite dimensional representation of $G$. If $A'/A$ is a finite extension of henselian discrete valuation rings such that $B' = A' \otimes_A B$ is a discrete valuation ring, then we have $\ar_n(\rho') = e_{A'/A} \ar_n(\rho)$, where $\rho'$ is $\rho$ viewed as a representation of the generic Galois group of the extension $B'/A'$.

Proof. For $\sigma \in G - \{1\}$, we have $I_{B'}(\sigma) = A' \otimes_A I_B(\sigma) = B' \otimes_B I_B(\sigma)$, so

$$i_{B'}(\sigma) = \text{length}_{B'}(B'/I_{B'}(\sigma)) = \text{length}_{B'}(B' \otimes_B B/I_B(\sigma)) = e_{B'/B} \text{length}_{B}(B/I_B(\sigma)) = e_{B'/B} \text{length}_{B}(I_{B'}(\sigma)).$$

Thus

$$\ar_n(\rho') = e_{B'/B} \ar_n(\rho') = e_{A'/A} \ar_n(\rho).$$

\[\square\]
Corollary 3. Let $B/A$ be a finite monogenic extension of henselian discrete valuation rings that is generically Galois with group $G$, and let $\rho : G \to \text{Aut}_C(V)$ be a finite dimensional representation of $G$. Then $\text{ar}_n(\rho)$ is an integer.

Proof. Restricting to the maximal unramified subextension of $B/A$ does not change the naive Artin conductor or the monogeneity of the extension. So assume $B/A$ is residually purely inseparable. Now just apply the previous proposition with $A'$ taken as in the first proposition and then use the classical Hasse-Arf theorem. $\square$

Remark. One can define a naive Swan conductor [1, 6.7] as well. It also is an integer in the monogenic case but simply because it agrees with the naive Artin conductor whenever $B/A$ is monogenic and not residually separable. It is not, however, a good invariant even in the monogenic case: it is a consequence of results outlined at the Luminy conference that in the (monogenic) equal-characteristic case, the naive Swan conductor of a faithful, one-dimensional representation agrees with Kato’s Swan conductor if and only if either $B/A$ is residually separable or $e_{B/A} = 1$, whereas for general monogenic extensions in equal-characteristic, the naive Artin conductor of a one-dimensional representation is equal to a non-logarithmic, “Artin-type” variant of Kato’s Swan conductor.

References


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