Dyadic diaphony of digital sequences

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1. Introduction

The diaphony $F_N$ (see [19] or [7, Definition 1.29] or [12, Exercise 5.27, p. 162]) of the first $N$ elements of a sequence $\omega = (x_n)_{n \geq 0}$ in $[0, 1)^s$ is given by

$$F_N(\omega) = \left( \sum_{\substack{k \in \mathbb{Z}^s \setminus \{0\} \atop k \neq 0}} \frac{1}{\rho(k)^2} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i \langle k, x_n \rangle} \right|^2 \right)^{1/2},$$

where for $k = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ it is $\rho(k) = \prod_{i=1}^{s} \max(1, |k_i|)$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^s$. It is well known that the diaphony is a quantitative measure for the irregularity of distribution of the first $N$
points of a sequence. In fact, a sequence \( \omega \) is uniformly distributed modulo 1 if and only if \( \lim_{N \to \infty} F_N(\omega) = 0 \). Throughout this paper we will call the diaphony the classical diaphony.

In [11] Hellekalek and Leeb introduced the notion of dyadic diaphony which is similar to the classical diaphony but with the trigonometric functions replaced by Walsh functions. Before we give the exact definition of the dyadic diaphony recall that Walsh-functions in base 2 can be defined as follows: for a non-negative integer \( k \) with base 2 representation \( k = \kappa_m 2^m + \cdots + \kappa_1 2 + \kappa_0 \) and a real \( x \) with (canonical) base 2 representation \( x = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots \) the \( k \)-th Walsh function in base 2 is defined as

\[
\operatorname{wal}_k(x) := (-1)^{x_1 \kappa_0 + x_2 \kappa_1 + \cdots + x_m \kappa_m}.
\]

For dimension \( s \geq 2 \), \( x_1, \ldots, x_s \in [0,1) \) and \( k_1, \ldots, k_s \in \mathbb{N}_0 \) we define

\[
\operatorname{wal}_{k_1, \ldots, k_s}(x_1, \ldots, x_s) := \prod_{j=1}^s \operatorname{wal}_{k_j}(x_j).
\]

For vectors \( k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \) and \( x = (x_1, \ldots, x_s) \in \mathbb{R}^s \) we write

\[
\operatorname{wal}_k(x) := \operatorname{wal}_{k_1, \ldots, k_s}(x_1, \ldots, x_s).
\]

Now we can give the definition of the dyadic diaphony (see Hellekalek and Leeb [11]).

**Definition.** The dyadic diaphony \( F_{2,N} \) of the first \( N \) elements of a sequence \( \omega = (x_n)_{n \geq 0} \) in \([0,1)^s\) is defined by

\[
F_{2,N}(\omega) = \left( \frac{1}{3^s - 1} \sum_{k \in \mathbb{N}_0^s} \frac{1}{\psi(k)^2} \left| \frac{1}{N} \sum_{n=0}^{N-1} \operatorname{wal}_k(x_n) \right|^2 \right)^{1/2},
\]

where for \( k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \) it is \( \psi(k) = \prod_{i=1}^s \psi(k_i) \) and for \( k \in \mathbb{N}_0 \),

\[
\psi(k) = \begin{cases} 1 & \text{if } k = 0, \\ 2^r & \text{if } 2^r \leq k < 2^{r+1} \text{ with } r \in \mathbb{N}_0. \end{cases}
\]

Throughout the paper we will write \( r(k) = r \) if \( r \) is the unique determined integer such that \( 2^r \leq k < 2^{r+1} \).

It is shown in [11, Theorem 3.1] that the dyadic diaphony is a quantitative measure for the irregularity of distribution of the first \( N \) points of a sequence: a sequence \( \omega \) is uniformly distributed modulo 1 if and only if \( \lim_{N \to \infty} F_{2,N}(\omega) = 0 \). Further it was shown in [5] that the dyadic diaphony is — up to a factor depending only on \( s \) — the worst-case error for quasi-Monte Carlo integration of functions from a certain Hilbert space.

We consider the dyadic diaphony of a special class of sequences in \([0,1)^s\), namely of so-called digital \((0,s)\)-sequences over \( \mathbb{Z}_2 \) for \( s = 1, 2 \). Digital
(0, s)-sequences or more generally digital (t, s)-sequences were introduced by Niederreiter [15, 16] and they provide at the moment the most efficient method to generate sequences with excellent distribution properties. We remark that a digital (0, s)-sequence over \( \mathbb{Z}_2 \) only exists if \( s = 1 \) or \( s = 2 \). For higher dimensions \( s \geq 3 \) the concept of digital (t, s)-sequence over \( \mathbb{Z}_2 \) with \( t > 0 \) has to be stressed (see [15] or [16]).

Before we give the definition of digital (0, s)-sequences we introduce some notation: for a vector \( \bar{c} = (\gamma_1, \gamma_2, \ldots) \in \mathbb{Z}_2^\infty \) and for \( m \in \mathbb{N} \) we denote the vector in \( \mathbb{Z}_2^m \) consisting of the first \( m \) components of \( \bar{c} \) by \( \bar{c}(m) \), i.e., \( \bar{c}(m) = (\gamma_1, \ldots, \gamma_m) \). Further for an \( \mathbb{N} \times \mathbb{N} \) matrix \( C \) over \( \mathbb{Z}_2 \) and for \( m \in \mathbb{N} \) we denote by \( C(m) \) the left upper \( m \times m \) submatrix of \( C \).

**Definition.** For \( s \in \{1, 2\} \), choose \( s \mathbb{N} \times \mathbb{N} \) matrices \( C_1, \ldots, C_s \) over \( \mathbb{Z}_2 \) with the following property: for every \( m \in \mathbb{N} \) and every \( 0 \leq n \leq m \) the vectors
\[
\bar{c}_1^{(1)}(m), \ldots, \bar{c}_n^{(1)}(m), \bar{c}_1^{(s)}(m), \ldots, \bar{c}_m^{-n}(m)
\]
are linearly independent in \( \mathbb{Z}_2^m \). Here \( \bar{c}_i^{(j)} \) is the \( i \)-th row vector of the matrix \( C_j \). (In particular for any \( m \in \mathbb{N} \) the matrix \( C_j(m) \) has full rank over \( \mathbb{Z}_2 \) for all \( j \in \{1, \ldots, s\} \).

For \( n \geq 0 \) let \( n = n_0 + n_1 2 + n_2 2^2 + \cdots \) be the base 2 representation of \( n \). For \( j \in \{1, \ldots, s\} \) multiply the vector \( \bar{n} = (n_0, n_1, \ldots)^T \) with the matrix \( C_j \),
\[
C_j \bar{n} =: (x_n^j(1), x_n^j(2), \ldots)^T \in \mathbb{Z}_2^\infty
\]
and set
\[
x_n^j := \frac{x_n^j(1)}{2} + \frac{x_n^j(2)}{2^2} + \cdots.
\]
Finally set \( x_n := (x_n^1, \ldots, x_n^s) \).

Every sequence \( (x_n)_{n \geq 0} \) constructed in this way is called digital (0, s)-sequence over \( \mathbb{Z}_2 \). The matrices \( C_1, \ldots, C_s \) are called the generator matrices of the sequence.

To guarantee that the points \( x_n \) belong to \( [0, 1]^s \) (and not just to \( [0, 1]^s \)) and also for the analysis of the sequence we need the condition that for each \( n \geq 0 \) and \( 1 \leq j \leq s \), we have \( x_n^j(i) = 0 \) for infinitely many \( i \). This condition is always satisfied if we assume that for each \( 1 \leq j \leq s \) and \( r \geq 0 \) we have \( c_{i,r}^j = 0 \) for all sufficiently large \( i \), where \( c_{i,r}^j \) are the entries of the matrix \( C_j \). Throughout this paper we assume that the generator matrices fulfill this condition (see [16, p.72] where this condition is called (S6)).

For example if \( s = 1 \) and if we choose as generator matrix the \( \mathbb{N} \times \mathbb{N} \) identity matrix, then the resulting digital (0, 1)-sequence over \( \mathbb{Z}_2 \) is the well known van der Corput sequence in base 2. Hence the concept of digital (0, 1)-sequences over \( \mathbb{Z}_2 \) is a generalization of the construction principle of the van der Corput sequence.
Note that finite versions of digital sequences over \( \mathbb{Z}_2 \) (so-called digital nets, see [16]) have a nice group structure, namely they are isomorphic to Cartesian products of the group \( \mathbb{Z}_2 \). The characters of these groups however are exactly the Walsh functions as defined above. For more information we refer to [13]. This is the reason why it is more convenient to consider the dyadic diaphony of digital sequences over \( \mathbb{Z}_2 \) instead of the classical diaphony. Furthermore this fact was used in many papers for computing different notions of discrepancies of digital point sets (see, for example, [2, 3, 4, 5, 6, 14, 17]).

For the classical diaphony it was proved by Faure [8] that

\[
(NF_N(\omega))^2 = \pi^2 \sum_{u=1}^{\infty} \left\| \frac{N}{2^u} \right\|^2,
\]

if \( \omega \) is a digital \((0,1)\)-sequence over \( \mathbb{Z}_2 \) whose generator matrix \( C \) is a non-singular upper triangular matrix. Faure (and we shall do so as well) called these sequences NUT-sequences. Here \( \| \cdot \| \) denotes the distance to the nearest integer function, i.e., \( \| x \| := \min(x - \lfloor x \rfloor, 1 - (x - \lfloor x \rfloor)) \). See also [1, 9, 10, 18] for further results concerning the classical diaphony of special 1-dimensional sequences.

The aim of this paper is to prove a similar formula for the dyadic diaphony of digital \((0,s)\)-sequences over \( \mathbb{Z}_2 \) for \( s \in \{1, 2\} \) (see Theorems 2.1 and 3.1). These formulae show that for fixed \( s \) the dyadic diaphony is invariant for digital \((0,s)\)-sequences over \( \mathbb{Z}_2 \). Further we find that the dyadic diaphony and the classical diaphony of NUT-sequences \((s = 1)\) only differ by a multiplicative constant (Corollary 2.2). We obtain the exact asymptotic order of the dyadic diaphony of digital \((0,s)\)-sequences over \( \mathbb{Z}_2 \) (Corollaries 2.3 and 3.2). Moreover it follows from our formula that the squared dyadic diaphony of digital \((0,1)\)-sequences over \( \mathbb{Z}_2 \) satisfies a central limit theorem (Corollary 2.4). For digital \((0,2)\)-sequences we will obtain a similar, but weaker result (Corollary 3.3).

### 2. The results for \( s = 1 \)

First we give the formula for the dyadic diaphony of digital \((0,1)\)-sequences over \( \mathbb{Z}_2 \). This formula shows that the dyadic diaphony is invariant for digital \((0,1)\)-sequences over \( \mathbb{Z}_2 \).

**Theorem 2.1.** Let \( \omega \) be a digital \((0,1)\)-sequence over \( \mathbb{Z}_2 \). Then for any \( N \geq 1 \) we have

\[
(NF_{N,N}(\omega))^2 = \pi^2 \sum_{u=1}^{\infty} \left\| \frac{N}{2^u} \right\|^2.
\]

We defer the proof of this formula to Section 4.
Remark. In Theorem 2.1 we have an infinite sum for the dyadic diaphony of a digital \((0,1)\)-sequence over \(\mathbb{Z}_2\). This formula can easily be made computable since for \(1 \leq N \leq 2^m\) we have \(\|N/2^u\| = N/2^u\) for \(u \geq m + 1\). Therefore we have

\[
(N F_{2,N}(\omega))^2 = 3 \sum_{u=1}^{m} \| N/2^u \|^2 + \left( \frac{N}{2^m} \right)^2.
\]  

From Theorem 2.1 we find the surprising result that the classical diaphony and the dyadic diaphony of a NUT-sequence are essentially the same.

**Corollary 2.2.** Let \(\omega\) be a NUT-sequence over \(\mathbb{Z}_2\). Then for any \(N \geq 1\) we have

\[
F_{2,N}(\omega) = \frac{\sqrt{3}}{\pi} F_N(\omega).
\]

**Proof.** This follows from Theorem 2.1 together with Faure’s formula (1). \(\square\)

From (2) one can see immediately that the dyadic diaphony of a digital \((0,1)\)-sequence over \(\mathbb{Z}_2\) is of order \(F_{2,N}(\omega) = O(\sqrt{\log N}/N)\). But we can even be much more precise. From a thorough analysis of the sum in (2) we obtain the exact dependence of the dyadic diaphony of digital \((0,1)\)-sequences over \(\mathbb{Z}_2\) on \(\sqrt{\log N}/N\).

**Corollary 2.3.** Let \(\omega\) be a digital \((0,1)\)-sequence over \(\mathbb{Z}_2\). For \(N \leq 2^m\) we have

\[
(N F_{2,N}(\omega))^2 \leq \frac{m}{3} + \frac{4}{3} - \frac{2(-1)^m}{9 \cdot 2^m} - \frac{1}{9 \cdot 2^{2m}}
\]

and

\[
\limsup_{N \to \infty} \frac{(N F_{2,N}(\omega))^2}{\log N} = \frac{1}{3 \log 2}.
\]

The proof of this result will be given in Section 5. We just remark that the result for the lim sup follows also from a result of Chaix and Faure [1, Théorème 4.13] for the classical diaphony of the van der Corput sequence together with Corollary 2.2 and Theorem 2.1.

In [6] it is shown that the star discrepancy and all \(L_p\)-discrepancies of the van der Corput sequence in base 2 satisfy a central limit theorem. The same arguments as in the proof of [6, Theorem 2] can now be used to obtain the subsequent result.

**Corollary 2.4.** Let \(\omega\) be a digital \((0,1)\)-sequence over \(\mathbb{Z}_2\). Then for every real \(y\) we have

\[
\frac{1}{M} \# \left\{ N < M : (N F_{2,N}(\omega))^2 \leq \frac{1}{4} \log_2 N + \frac{1}{4\sqrt{3}} \sqrt{\log_2 N} \right\} = \Phi(y) + o(1),
\]
where
\[ \Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}} dt \]
denotes the normal distribution function and \( \log_2 \) denotes the logarithm to the base 2. I.e., the squared dyadic diaphony of a digital \((0,1)\)-sequence over \(\mathbb{Z}_2\) satisfies a central limit theorem.

**Remark.** Together with Corollary 2.2 we also obtain a central limit theorem for the square of the classical diaphony of NUT-sequences.

**Proof.** As already mentioned, the proof follows exactly the lines of the proof of [6, Theorem 2]. One only has to compute the expectation and the variance of the random variable
\[ S_m = \sum_{w=1}^{m} \|X2^w\|^2, \]
where \( X \) is uniformly distributed on \([0,1)\). By tedious but straightforward calculations we obtain \( \mathbb{E}S_m = m/12 \) and \( \text{Var}S_m = m/432 + 7(1 - 2^{-2m})/1620. \)

3. The results for \( s = 2 \)

We give the formula for the dyadic diaphony of digital \((0,2)\)-sequences over \(\mathbb{Z}_2\) which shows that the dyadic diaphony is invariant for digital \((0,2)\)-sequences as well.

**Theorem 3.1.** Let \( \omega \) be a digital \((0,2)\)-sequence over \(\mathbb{Z}_2\). Then for any \( N \geq 1 \) we have
\[ (NF_{2,N}(\omega))^2 = \frac{9}{4} \sum_{u=1}^{\infty} \left\| \frac{N}{2^u} \right\|^2 u. \]

We defer the proof of this formula to Section 4.

**Remark.** In Theorem 3.1 we have an infinite sum for the dyadic diaphony of a digital \((0,2)\)-sequence over \(\mathbb{Z}_2\). Again this formula can easily be made computable. For \( 1 \leq N \leq 2^m \) we have
\[ (NF_{2,N}(\omega))^2 = \frac{9}{4} \sum_{u=1}^{m} \left\| \frac{N}{2^u} \right\|^2 u + \left( \frac{N}{2^m} \right)^2 \frac{4 + 3m}{4}. \]

From (3) one can see immediately that the dyadic diaphony of a digital \((0,2)\)-sequence over \(\mathbb{Z}_2\) is of order \( F_{2,N}(\omega) = O(\log N/N) \). Also here we obtain from a thorough analysis of the sum in (3) the exact dependence of the dyadic diaphony of digital \((0,2)\)-sequences over \(\mathbb{Z}_2\) on \( \log N/N \).
Corollary 3.2. Let \( \omega \) be a digital \((0, 2)\)-sequence over \( \mathbb{Z}_2 \). Then for any \( N \leq 2^m \) we have

\[
(NF_{2,N}(\omega))^2 \leq \frac{m^2}{8} + \frac{7m}{8} + \frac{11}{9} + O\left(\frac{m}{2^m}\right)
\]

and

\[
\limsup_{N \to \infty} \frac{(NF_{2,N}(\omega))^2}{(\log N)^2} = \frac{1}{8(\log 2)^2}.
\]

The proof of this result will be given in Section 5. Following this proof the \( O(m/2^m) \)-term in the above bound can easily be made explicit.

Unfortunately we could not show that the squared dyadic diaphony of a digital \((0, 2)\)-sequence over \( \mathbb{Z}_2 \) satisfies a central limit theorem. However, we were able to prove the following result.

Corollary 3.3. Let \( \omega \) be a digital \((0, 2)\)-sequence over \( \mathbb{Z}_2 \). Then for any \( \varepsilon > 0 \) we have

\[
\lim_{m \to \infty} \frac{1}{2^m} \# \left\{ N < 2^m : \frac{3}{32} - \varepsilon < \left( \frac{NF_{2,N}(\omega)}{\log_2 N} \right)^2 < \frac{3}{32} + \varepsilon \right\} = 1.
\]

Proof. By tedious but straightforward calculations using Theorem 3.1 we obtain

\[
\sum_{N=0}^{2^m-1} (NF_{2,N}(\omega))^2 = \frac{3}{32} m^2 2^m + O(m2^m)
\]

and

\[
\sum_{N=0}^{2^m-1} (NF_{2,N}(\omega))^4 = \frac{9}{1024} m^4 2^m + O(m^3 2^m).
\]

From this the result immediately follows. \( \square \)

4. The proofs of Theorems 2.1 and 3.1

For the proofs of Theorems 2.1 and 3.1 we need the subsequent lemma. This result was implicitly proved in [6]. For the sake of completeness we provide the short proof.

Lemma 4.1. Let the non-negative integer \( U \) have binary expansion \( U = U_0 + U_12 + \cdots + U_{m-1}2^{m-1} \). For any non-negative integer \( n \leq U - 1 \) let \( n = n_0 + n_12 + \cdots + n_{m-1}2^{m-1} \) be the binary representation of \( n \). For \( 0 \leq p \leq m-1 \) let \( U(p) := U_0 + \cdots + U_p2^p \). Let \( b_0, b_1, \ldots, b_{m-1} \) be arbitrary elements of \( \mathbb{Z}_2 \), not all zero. Then

\[
\sum_{n=0}^{U-1} (-1)^{b_0n_0 + \cdots + b_{m-1}n_{m-1}} = (-1)^{b_{w+1}U_{w+1} + \cdots + b_{m-1}U_{m-1}2^w} \left\| \frac{U}{2^w} \right\|,
\]
Proof. From splitting up the sum we obtain

\[
\sum_{n=0}^{U-1} (-1)^{b_0n_0+\cdots+b_{m-1}n_{m-1}} 2^{w+1}(U_{w+1}+\cdots+U_{m-1}2^{m-w-2})^{-1} = \sum_{n=0}^{U(w)-1} (-1)^{n_w} (-1)^{b_{w+1}n_{w+1}+\cdots+b_{m-1}n_{m-1}} \\
+ \sum_{n=0}^{U(w)-1} (-1)^{n_w} (-1)^{b_{w+1}U_{w+1}+\cdots+b_{m-1}U_{m-1}} \\
= 0 + (-1)^{b_{w+1}U_{w+1}+\cdots+b_{m-1}U_{m-1}} U(w)-1 \sum_{n=0}^{U(w)-1} (-1)^{n_w} \\
= (-1)^{b_{w+1}U_{w+1}+\cdots+b_{m-1}U_{m-1}} \times \left\{ \begin{array}{ll}
U(w) & \text{if } U(w) < 2^w, \\
2^{w+1} - U(w) & \text{if } U(w) \geq 2^w,
\end{array} \right.
= (-1)^{b_{w+1}U_{w+1}+\cdots+b_{m-1}U_{m-1}} 2^{w+1} \times \left\{ \begin{array}{ll}
\frac{U(w)}{2^{w+1}} & \text{if } \frac{U(w)}{2^{w+1}} < \frac{1}{2}, \\
1 - \frac{U(w)}{2^{w+1}} & \text{if } \frac{U(w)}{2^{w+1}} \geq \frac{1}{2},
\end{array} \right.
= (-1)^{b_{w+1}U_{w+1}+\cdots+b_{m-1}U_{m-1}} 2^{w+1} \left\| \frac{U(w)}{2^{w+1}} \right\|.
\]

Since \( \left\| \frac{U(w)}{2^{w+1}} \right\| = \left\| \frac{U}{2^{w+1}} \right\| \) the result follows. \(\square\)

Now we can give the

Proof of Theorem 2.1. Let \( 2^r \leq k < 2^{r+1} \). Then \( k = k_0 + k_12 + \cdots + k_r2^r \) with \( k_i \in \{0,1\}, 0 \leq i < r \) and \( k_r = 1 \). Let \( \langle \cdot, \cdot \rangle \) denote the usual inner product in \( \mathbb{Z}_{2^r}^r \) and let \( \vec{c}_i \in \mathbb{Z}_{2^r}^\infty \) be the \( i \)-th row vector of the generator matrix \( C \) of the digital \((0,1)\)-sequence (for short we write \( C \) instead of \( C_1 \) here). Since the \( i \)-th digit \( x_n(i) \) of the point \( x_n, i \in \mathbb{N}, n \in \mathbb{N}_0 \), is given by \( \langle \vec{c}_i, \vec{n} \rangle \) (see Definition 1) we have

\[
\sum_{n=0}^{N-1} \text{walk}(x_n) = \sum_{n=0}^{N-1} (-1)^{k_0\langle \vec{c}_1, \vec{n} \rangle + \cdots + k_r\langle \vec{c}_{r+1}, \vec{n} \rangle} \\
= \sum_{n=0}^{N-1} (-1)^{k_0\vec{c}_1 + \cdots + k_r\vec{c}_{r+1}, \vec{n}}. \tag{4}
\]

Let \( C = (c_{i,j})_{i,j \geq 1} \). For \( k \in \mathbb{N}, k = k_0 + k_12 + \cdots + k_r2^r, k_i \in \{0,1\}, 0 \leq i < r \) and \( k_r = 1 \) define \( u(k) := \min\{l \geq 1 : k_0c_{1,l} + \cdots + k_rc_{r+1,l} = 1\} \).

Note that since \( C \) generates a digital \((0,1)\)-sequence over \( \mathbb{Z}_2 \) we obviously have \( u(k) \leq r + 1 \). For fixed \( k, 2^r \leq k < 2^{r+1} \) let \( \vec{b} = (b_0, b_1, \ldots)^\top := \)
the number of vectors \( (k_0, \ldots, k_r) \). Let \( N = N_0 + N_1 2 + \cdots + N_{m-1} 2^{m-1} \). If \( u(k) \leq m \) we obtain from (4) together with Lemma 4.1,

\[
N_1(z) = \sum_{n=0}^{N-1} (-1)^{(b,b)} = \sum_{n=0}^{N-1} (-1)^{n_0 b_0 + \cdots + n_{m-1} b_{m-1}} = \sum_{n=0}^{N-1} (-1)^{n_u(k) b_u(k) + \cdots + n_u(k) b_u(k)} \frac{N}{2^{u(k)}}.
\]

But if \( u(k) > m \) the above equality is trivially true. Therefore we have

\[
2(NF_{2,N}(\omega))^2 = \sum_{k=1}^{\infty} \frac{1}{2^{2u(k)}} \left( \frac{N}{2^{u(k)}} \right)^2 = \sum_{r=0}^{\infty} \frac{1}{2^{2r}} \sum_{k=2^r}^{2^{r+1}-1} 2^{2u(k)} \left( \frac{N}{2^{u(k)}} \right)^2 = \sum_{r=0}^{\infty} \frac{1}{2^{2r}} \sum_{u=1}^{r+1} 2^{2u} \left( \frac{N}{2^{u}} \right)^2 \sum_{u(k)=u}^{2^{r+1}-1} 2^{u(k)} \sum_{u(k)=u}^{2^{r+1}-1} 1.
\]

Now we have to evaluate the sum \( \sum_{u(k)=u}^{2^{r+1}-1} 1 \) for \( r \geq u - 1 \) and \( u \geq 1 \). This is the number of vectors \( (k_0, \ldots, k_{r-1})^T \in \mathbb{Z}_2^r \) such that

\[
(5) \quad C(r + 1)^T \begin{pmatrix}
  k_0 \\
  \vdots \\
  k_{r-1} \\
  1
\end{pmatrix} = \begin{pmatrix}
  0 \\
  \vdots \\
  0 \\
  1 \\
  x_{u+1} \\
  \vdots \\
  x_{r+1}
\end{pmatrix} \in \mathbb{Z}_2^{r+1}
\]

for arbitrary \( x_{u+1}, \ldots, x_{r+1} \in \mathbb{Z}_2 \). (Recall that for an integer \( m \geq 1 \) we denote by \( C(m) \) the left upper \( m \times m \) submatrix of the matrix \( C \), see Section 1.)

We consider two cases:
(i) Assume that \( r = u - 1 \). Then system (5) becomes

\[
C(r + 1)^\top \begin{pmatrix}
k_0 \\
\vdots \\
k_{r-1} \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix}.
\]

Since the \((r + 1) \times (r + 1)\) matrix \( C(r + 1)^\top \) is regular over \( \mathbb{Z}_2 \) it is clear that there exists a vector \( \vec{k} = (k_0, \ldots, k_r) \in \mathbb{Z}_2^{r+1} \) such that \( C(r + 1)^\top \vec{k} = (0, \ldots, 0, 1)^\top \). Assume that \( k_r = 0 \), then we have \( C(r)^\top (k_0, \ldots, k_{r-1})^\top = (0, \ldots, 0)^\top \). Again we know that \( C(r)^\top \) is regular over \( \mathbb{Z}_2 \) and therefore we obtain \( k_0 = \cdots = k_{r-1} = 0 \). Hence \( \vec{k} = \vec{0} \), the zero vector in \( \mathbb{Z}_2^{r+1} \). This is now a contradiction since \( \vec{k} \) is a solution of the system \( C(r + 1)^\top \vec{k} = (0, \ldots, 0, 1)^\top \). Therefore we have

\[
\sum_{k=2^u-1 \atop u(k) = u} 1 = 1.
\]

(ii) Assume that \( r \geq u \). Since \( C(r) \) is regular over \( \mathbb{Z}_2 \) it is clear that \( D(r) := (C(r)^\top)^{-1} \) is regular over \( \mathbb{Z}_2 \). Hence for any vector \( \vec{k} \in \mathbb{Z}_2^r \) there is a vector \( \vec{l} \in \mathbb{Z}_2^r \) such that \( \vec{k} = D(r)\vec{l} \). Therefore system (5) can be rewritten as

\[
C(r + 1)^\top \begin{pmatrix}
D(r)\vec{l} \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1 \\
x_{u+1} \\
\vdots \\
x_{r+1}
\end{pmatrix}
\]

with \( \vec{l} \in \mathbb{Z}_2^r \). Now we use the definition of the matrix \( D(r) \) and find that the above system is equivalent to the system

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots \\
0 & 0 & \ldots & 1 & 0 \\
d_1 & d_2 & \ldots & d_{r-1} & d_r
\end{pmatrix}
\begin{pmatrix}
l_0 \\
\vdots \\
l_{r-1}
\end{pmatrix}
= \begin{pmatrix}
0 \\
\vdots \\
0 \\
x_{u+1} \\
\vdots \\
x_{r+1}
\end{pmatrix}
+ \vec{c}_{r+1}(r + 1)^\top,
\]
where \((d_1, \ldots, d_r) := (c_{1,r+1}, \ldots, c_{r,r+1})D(r)\). Now one can easily see that for arbitrary \(x_u, \ldots, x_r\) there exists exactly one solution \(\bar{l} = (l_0, \ldots, l_{r-1})^\top \in \mathbb{Z}_2^r\) such that the first \(r\) lines of the above system are fulfilled. Further there is exactly one possible choice of \(x_{r+1} \in \mathbb{Z}_2\) such that this vector \(\bar{l}\) is a solution of the above system. Therefore we obtain
\[
\sum_{k=2^r}^{2^{r+1}-1} 1 = 2^{r-u}.
\]

Now we have
\[
2(NF_{2,N}(\omega))^2 = \sum_{u=1}^{\infty} \left\| \frac{N}{2^n} \right\|^2 2^{2u} \left( \frac{1}{2^{2u-1}} + \sum_{r=u}^{\infty} \frac{1}{2^r} 2^{r-u} \right) = 6 \sum_{u=1}^{\infty} \left\| \frac{N}{2^n} \right\|^2.
\]

The result follows. □

Proof of Theorem 3.1. Let \(\omega = (x_n)_{n \geq 0}\) be a digital \((0, 2)\)-sequence over \(\mathbb{Z}_2\). Let \(x_n = (x_n, y_n)\) for \(n \geq 0\). Clearly the sequences \((x_n)_{n \geq 0}\) and \((y_n)_{n \geq 0}\) are digital \((0,1)\)-sequences over \(\mathbb{Z}_2\). We have
\[
(NF_{2,N}(\omega))^2 = \frac{1}{8} \sum_{k \in \mathbb{Z}_2^*} \frac{1}{\mu(k)} \sum_{n=0}^{N-1} \left| \text{wal}_k(x_n) \right|^2
\]
\[
= \frac{1}{8} \sum_{k=1}^{\infty} \frac{1}{2^{2r(k)}} \sum_{n=0}^{N-1} \left| \text{wal}_k(x_n) \right|^2 + \frac{1}{8} \sum_{l=1}^{\infty} \frac{1}{2^{2r(l)}} \sum_{n=0}^{N-1} \left| \text{wal}_l(y_n) \right|^2
\]
\[
+ \frac{1}{8} \sum_{k,l=1}^{\infty} \frac{1}{2^{2r(k)+2r(l)}} \sum_{n=0}^{N-1} \left| \text{wal}_k(x_n) \text{wal}_l(y_n) \right|^2
\]
\[
= \frac{3}{2} \sum_{u=1}^{\infty} \left\| \frac{N}{2^n} \right\|^2 + \frac{1}{8} \sum_{k,l=1}^{\infty} \frac{1}{2^{2r(k)+2r(l)}} \sum_{n=0}^{N-1} \left| \text{wal}_k(x_n) \text{wal}_l(y_n) \right|^2,
\]
where for the last equality we used Theorem 2.1. We have to consider
\[
\Sigma := \sum_{k,l=1}^{\infty} \frac{1}{2^{2r(k)+2r(l)}} \sum_{n=0}^{N-1} \left| \text{wal}_k(x_n) \text{wal}_l(y_n) \right|^2.
\]
Assume that \(2^r \leq k < 2^{r+1}\) and \(2^t \leq l < 2^{t+1}\). Then \(k = k_0 + k_1 2 + \cdots + k_r 2^r\) with \(k_i \in \{0, 1\}, 0 \leq i < r\) and \(k_r = 1\) and \(l = l_0 + l_1 2 + \cdots + l_t 2^t\) with \(l_j \in \{0, 1\}, 0 \leq j < t\) and \(l_t = 1\). Let \(\vec{c_i} \in \mathbb{Z}_2^n\) be the \(i\)-th row vector of the generator matrix \(C_1\) and let \(\vec{d_i} \in \mathbb{Z}_2^n\) be the \(i\)-th row vector of the generator matrix \(C_2\), \(i \in \mathbb{N}\). Since the \(i\)-th digit \(x_n(i)\) of \(x_n\) is given by \((\vec{c_i}, \vec{n})\) and the
Note that since

\[ C \]

\[ \text{To this end we define the (r, t)} \]

\[ \text{Therefore we have} \]

\[ \sum_{n=0}^{N-1} \text{wal}_k(x_n) \text{wal}_l(y_n) = \sum_{n=0}^{N-1} \left( -1 \right)^{k_0(c_1, n) + \cdots + k_r(c_{r+1}, n) + l_0(d_1, n) + \cdots + l_t(d_{t+1}, n)} \]

\[ = \sum_{n=0}^{N-1} \left( -1 \right)^{k_0(c_1, n) + \cdots + k_r(c_{r+1}, n) + l_0(d_1, n) + \cdots + l_t(d_{t+1}, n)} \]

\[ \text{Let } C_1 = (c_{i,j})_{i,j \geq 1} \text{ and } C_2 = (d_{i,j})_{i,j \geq 1}. \]

\[ \text{Define} \]

\[ u(k, l) := \min \{ j \geq 1 : k_0 c_{1,j} + \cdots + k_r c_{r+1,j} + l_0 d_{1,j} + \cdots + l_t d_{t+1,j} = 1 \} \]

\[ \text{Since } C_1, C_2 \text{ generate a digital (0, 2)-sequence over } \mathbb{Z}_2 \text{ we obviously have} \]

\[ u(k, l) \leq r + t + 2. \]

\[ \text{As in the proof of Theorem 2.1 we now apply Lemma 4.1 and obtain} \]

\[ \left| \sum_{n=0}^{N-1} \text{wal}_k(x_n) \text{wal}_l(y_n) \right| = 2^{u(k, l)} \left| \frac{N}{2^{u(k, l)}} \right|^2. \]

\[ \text{Therefore we have} \]

\[ \Sigma = \sum_{k,l=1}^{\infty} \frac{1}{2^{2r(k) + 2r(l) + 2^{u(k, l)}}} \left| \frac{N}{2^{u(k, l)}} \right|^2 \]

\[ = \sum_{r,t=0}^{\infty} \frac{1}{2^{2r+2t}} \sum_{k=2^r}^{2^{r+1}} \sum_{l=2^t}^{2^{t+1} - 1} 2^{u(k, l)} \left| \frac{N}{2^{u(k, l)}} \right|^2 \]

\[ = \sum_{r,t=0}^{\infty} \frac{1}{2^{2r+2t}} \sum_{u=1}^{r+t+2} 2^{u} \left| \frac{N}{2^{u}} \right| \sum_{k=2^r}^{2^{r+1}} \sum_{l=2^t}^{2^{t+1} - 1} 1. \]

\[ \text{We have to evaluate the double-sum} \]

\[ \sum_{k=2^r}^{2^{r+1}} \sum_{l=2^t}^{2^{t+1} - 1} 1 \text{ for } 1 \leq u \leq r + t + 2. \]

\[ \text{To this end we define the (r + t + 2) \times (r + t + 2) matrix} \]

\[ \mathcal{C}(r, t) := \begin{pmatrix} c_{1,1} & \cdots & c_{r+1,1} & d_{1,1} & \cdots & d_{t+1,1} \\ c_{1,2} & \cdots & c_{r+1,2} & d_{1,2} & \cdots & d_{t+1,2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{1,r+t+2} & \cdots & c_{r+1,r+t+2} & d_{1,r+t+2} & \cdots & d_{t+1,r+t+2} \end{pmatrix}. \]

\[ \text{Note that since } C_1, C_2 \text{ generate a digital (0, 2)-sequence over } \mathbb{Z}_2, \text{ it follows that} \]

\[ \mathcal{C}(r, t) \text{ is regular.} \]
Now the value of the above double-sum is exactly the number of digits $k_0, \ldots, k_r, l_0, \ldots, l_{t-1} \in \mathbb{Z}_2$ such that

$$\mathcal{C}(r, t) \begin{pmatrix} k_0 \\ \vdots \\ k_{r-1} \\ 1 \\ l_0 \\ \vdots \\ l_{t-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ x_{u+1} \\ \vdots \\ x_{r+t+2} \end{pmatrix}$$

for arbitrary $x_{u+1}, \ldots, x_{r+t+2} \in \mathbb{Z}_2$. We consider three cases:

(i) Assume that $u = r + t + 2$. Then system (7) becomes

$$\mathcal{C}(r, t) \vec{h} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{8}$$

Since $\mathcal{C}(r, t)$ is regular there exists a vector $\vec{h} = (k_0, \ldots, k_r, l_0, \ldots, l_{t})^\top \in \mathbb{Z}_2^{r+t+2}$, $\vec{h} \neq \vec{0}$, such that $\mathcal{C}(r, t)\vec{h} = (0, \ldots, 0, 1)^\top$. Assume that $l_t = 0$. Then

$$\begin{pmatrix} c_{1,1} & \cdots & c_{r+1,1} & d_{1,1} & \cdots & d_{t,1} \\ c_{1,2} & \cdots & c_{r+1,2} & d_{1,2} & \cdots & d_{t,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{1,r+t+2} & \cdots & c_{r+1,r+t+2} & d_{1,r+t+2} & \cdots & d_{t,r+t+2} \end{pmatrix} \begin{pmatrix} k_0 \\ \vdots \\ k_r \\ 1 \\ l_0 \\ \vdots \\ l_{t-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{9}$$

But then

$$\begin{pmatrix} c_{1,1} & \cdots & c_{r+1,1} & d_{1,1} & \cdots & d_{t,1} \\ c_{1,2} & \cdots & c_{r+1,2} & d_{1,2} & \cdots & d_{t,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{1,r+t+1} & \cdots & c_{r+1,r+t+1} & d_{1,r+t+1} & \cdots & d_{t,r+t+1} \end{pmatrix} \begin{pmatrix} k_0 \\ \vdots \\ k_r \\ 1 \\ l_0 \\ \vdots \\ l_{t-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{10}$$

Since the above matrix is again regular we obtain that the vector $(k_0, \ldots, k_r, l_0, \ldots, l_{t-1}) = (0, \ldots, 0)$ and therefore $\vec{h} = \vec{0}$, a contradiction. Hence $l_t = 1$ and in the same way one can show that $k_r = 1$. We
have shown that system (8) has exactly one solution and therefore we have
\[ \sum_{k=2^r}^{2^{r+1}-1} \sum_{l=2^t}^{2^{t+1}-1} \frac{1}{u(k,l)=u} = 1. \]

(ii) Assume that \( u = r + t + 1 \). Let \( x \in \mathbb{Z}_2 \). Since \( C(r,t) \) is regular there exists exactly one vector \( \vec{h} \in \mathbb{Z}_{2}^{r+t+2} \) such that
\[ C(r,t)\vec{h} = (0, \ldots, 0, 1, x)^\top. \]
Assume that \( \vec{h} \) is of the form \( \vec{h} = (k_0, \ldots, k_{r-1}, 1, l_0, \ldots, l_{t-1}, 1)^\top \in \mathbb{Z}_2^{r+t+2} \). In particular we have
\[
\begin{pmatrix}
  c_{1,1} & \cdots & c_{r,1} & d_{1,1} & \cdots & d_{t,1} \\
  c_{1,2} & \cdots & c_{r,2} & d_{1,2} & \cdots & d_{t,2} \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  c_{1, r+t} & \cdots & c_{r, r+t} & d_{1, r+t} & \cdots & d_{t, r+t}
\end{pmatrix}
\begin{pmatrix}
  0 \\
  \vdots \\
  0
\end{pmatrix} = \begin{pmatrix}
  0 \\
  \vdots \\
  0
\end{pmatrix} + y_{r,t}
\]

(iii) Assume that \( 1 \leq u \leq r + t \). We rewrite system (7) in the form
\[
\begin{pmatrix}
  c_{1,1} & \cdots & c_{r,1} & d_{1,1} & \cdots & d_{t,1} \\
  c_{1,2} & \cdots & c_{r,2} & d_{1,2} & \cdots & d_{t,2} \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  c_{1, r+t} & \cdots & c_{r, r+t} & d_{1, r+t} & \cdots & d_{t, r+t}
\end{pmatrix}
\begin{pmatrix}
  k_0 \\
  \vdots \\
  k_{r-1} \\
  l_0 \\
  \vdots \\
  l_{t-1}
\end{pmatrix} = \begin{pmatrix}
  0 \\
  \vdots \\
  1 \\
  x_{u+1} \\
  \vdots \\
  x_{r+t+1}
\end{pmatrix} + y_{r,t}
\]
where \( \bar{y}_{r,t} = (c_{r+1,1} + d_{t+1,1}, \ldots, c_{r+1,r+t+2} + d_{t+1,r+t+2})^\top \in \mathbb{Z}_{2}^{r+t+2} \).

Since the upper \((r + t) \times (r + t)\) sub-matrix of the above matrix is regular we find for arbitrary \(x_{u+1}, \ldots, x_{r+t}\) exactly one solution of the first \(r + t\) rows of the above system. This solution can be made a solution of the whole system by an adequate choice of \(x_{r+t+1}\) and \(x_{r+t+2}\). Therefore we have

\[
\sum_{k=2^r}^{2^{r+1}-1} \sum_{l=2^t}^{2^{t+1}-1} 1 = 2^{r+t-u}.
\]

Now we have

\[
\Sigma = \sum_{r,t=0}^{\infty} \frac{1}{2^{2r+2t}} \left( \sum_{u=1}^{r+t} 2^{2u} \left\| \frac{N}{2^u} \right\|^2 2^{r+t-u} + 2^{2(r+t+2)} \left\| \frac{N}{2^{r+t+2}} \right\|^2 \right).
\]

\[
= \sum_{r,t=0}^{\infty} \frac{1}{2^{2r+2t}} \sum_{u=1}^{r+t} 2^{2u} \left\| \frac{N}{2^u} \right\|^2 + 16 \sum_{r,t=0}^{\infty} \sum_{u=2}^{\infty} \left\| \frac{N}{2^u} \right\|^2 \sum_{r+t=u-2}^{1} 1
\]

\[
= \sum_{u=1}^{\infty} 2^u \left\| \frac{N}{2^u} \right\|^2 \sum_{u=2}^{\infty} \frac{1}{2^{r+t}} + 16 \sum_{u=2}^{\infty} \left\| \frac{N}{2^u} \right\|^2 (u - 1)
\]

\[
= \sum_{u=1}^{\infty} \left\| \frac{N}{2^u} \right\|^2 (2u + 4) + 16 \sum_{u=2}^{\infty} \left\| \frac{N}{2^u} \right\|^2 (u - 1)
\]

\[
= \sum_{u=1}^{\infty} \left\| \frac{N}{2^u} \right\|^2 (18u - 12).
\]

The result follows by inserting this expression into (6). □

5. The proofs of Corollaries 2.3 and 3.2

We will say that a real \( \beta \) in \([0, 1)\) is \( m \)-bit if \( \beta = b_1 2^{-1} + \cdots + b_m 2^{-m} \) with \( b_i \in \{0, 1\} \). I.e., an \( m \)-bit number is of the form \( k/2^m \) with \( k \in \{0, 1, \ldots, 2^m - 1\} \).

The essential technical tool for the proof of Corollary 2.3 is provided by

Lemma 5.1. Assume that \( \beta = 0.b_1 b_2 \ldots \) (this here and in the following always means base 2 representation) has two equal consecutive digits \( b_i b_{i+1} \)
with \( i \leq m - 1 \) and let \( i \) be minimal with this property, i.e.,

\[
\beta = 0, 01 \ldots 0100b_i+2 \ldots \quad \text{or} \\
\beta = 0, 10 \ldots 0100b_i+2 \ldots \quad \text{or} \\
\beta = 0, 01 \ldots 101b_i+2 \ldots \quad \text{or} \\
\beta = 0, 10 \ldots 101b_i+2 \ldots .
\]

Replace \( \beta \) by

\[
\gamma = 0, 01 \ldots 0100b_i+2 \ldots \quad \text{resp.} \\
\gamma = 0, 10 \ldots 010b_i+2 \ldots \quad \text{resp.} \\
\gamma = 0, 01 \ldots 101b_i+2 \ldots \quad \text{resp.} \\
\gamma = 0, 01 \ldots 0101b_i+2 \ldots .
\]

Then

\[
\sum_{u=0}^{m-1} \|2^u \gamma\|^2 = \sum_{u=0}^{m-1} \|2^u \beta\|^2 + \begin{cases} 
\frac{1}{9} \left( 1 - \frac{(-1)^i}{2^i} \right)^2 (1 - \tau) & \text{in the first two cases,} \\
\frac{1}{9} \left( 1 - \frac{(-1)^i}{2^i} \right)^2 \tau & \text{in the last two cases,}
\end{cases}
\]

where \( \tau := 0, b_i+2b_{i+3} \ldots . \)

Remark. In any case we have \( \sum_{u=0}^{m-1} \|2^u \gamma\|^2 \geq \sum_{u=0}^{m-1} \|2^u \beta\|^2 \) with equality iff \( \tau = 1 \) in the first two cases and iff \( \tau = 0 \) in the last two cases.

**Proof of Lemma 5.1.** This is simple calculation. We just handle the first case here.

\[
\sum_{u=0}^{m-1} \left( \|2^u \gamma\|^2 - \|2^u \beta\|^2 \right) = \|\gamma\|^2 - \|2^i \beta\|^2 + \sum_{u=0}^{i-1} (\|2^u (2\gamma)\|^2 - \|2^u \beta\|^2).
\]

Here \( \|\gamma\| = \frac{1}{3} \left( 1 + \frac{1}{2^i} \right) - \frac{\tau}{2^{i+1}} \) and \( \|2^i \beta\| = \frac{\tau}{2^{i+1}} \). Further, for even \( u \) we have

\[
\|2^u (2\gamma)\| = \frac{1}{3} \left( 1 - \frac{2^{u+1}}{2^i} \right) + \frac{\tau}{2^{i-u}} \quad \text{and} \quad \|2^u \beta\| = \frac{1}{3} \left( 1 - \frac{2^{u+1}}{2^i} \right) + \frac{\tau}{2^{i+1-u}},
\]

and for odd \( u \) we have

\[
\|2^u (2\gamma)\| = \frac{1}{3} \left( 1 + \frac{2^{u+1}}{2^i} \right) - \frac{\tau}{2^{i-u}} \quad \text{and} \quad \|2^u \beta\| = \frac{1}{3} \left( 1 + \frac{2^{u+1}}{2^i} \right) - \frac{\tau}{2^{i+1-u}}.
\]

Inserting this into (10) we obtain

\[
\sum_{u=0}^{m-1} (\|2^u \gamma\|^2 - \|2^u \beta\|^2) = \frac{1}{9} \left( 1 + \frac{1}{2^i} \right)^2 (1 - \tau).
\]

The other cases are calculated in the same way. \( \Box \)
From Lemma 5.1 we obtain the subsequent result concerning the maximum of \( \sum_{u=0}^{m-1} \|2^u \beta \|^2 \) over all \( \beta \). We remark that in [14] the authors considered the same problem without the square at the \( \| \cdot \| \)-function.

**Lemma 5.2.** Consider \( \beta \in \mathbb{R} \) with the canonical base 2 representation (i.e., with infinitely many digits equal to zero). Then there exists

\[
\max_{\beta} \sum_{u=0}^{m-1} \|2^u \beta \|^2 = \frac{m}{9} + \frac{1}{9} - (-1)^m \frac{2}{27 \cdot 2^m} - \frac{1}{27 \cdot 2^{2m}}
\]

and it is attained if and only if \( \beta \) is of the form \( \beta_0 \) with

\[
\beta_0 = \frac{2}{3} \left( 1 - \left( -\frac{1}{2} \right)^{m+1} \right) \quad \text{or} \quad \beta_0 = \frac{1}{3} \left( 1 - \left( -\frac{1}{2} \right)^{m} \right).
\]

**Remark.** Note that

\[
\frac{2}{3} \left( 1 - \left( \frac{1}{2} \right)^{m+1} \right) = \begin{cases} 
0,1010\ldots101 & \text{if } m \text{ is odd}, \\
0,1010\ldots011 & \text{if } m \text{ is even},
\end{cases}
\]

and

\[
\frac{1}{3} \left( 1 - \left( -\frac{1}{2} \right)^{m} \right) = \begin{cases} 
0,0101\ldots011 & \text{if } m \text{ is odd}, \\
0,0101\ldots101 & \text{if } m \text{ is even}.
\end{cases}
\]

**Proof of Lemma 5.2.** For any \( \gamma = 0, c_1 c_2 \ldots c_{m} c_{m+1} \ldots \) with fixed \( c_1, \ldots, c_m \), the sum \( \sum_{u=0}^{m-1} \|2^u \gamma \|^2 \) obviously becomes maximal if \( c_m = 0 \) and \( c_{m+1} = c_{m+2} = \cdots = 1 \), or if \( c_m = 1 \) and \( c_{m+1} = c_{m+2} = \cdots = 0 \). Hence by Lemma 5.1 the

\[
\sup_{\beta} \sum_{u=0}^{m-1} \|2^u \beta \|^2
\]

only can be attained, respectively approached by

- \( \beta_1 = 0, 1010\ldots10111\ldots \) or
  \( (b_m \text{ is the last zero}) \)
- \( \beta_2 = 0, 0101\ldots01 \) or
- \( \beta_3 = 0, 1010\ldots11 \)
  \( (b_m \text{ is the last one}) \)

if \( m \) is even, and by

- \( \beta_4 = 0, 0101\ldots10111\ldots \) or
  \( (b_m \text{ is the last zero}) \)
- \( \beta_5 = 0, 1010\ldots01 \) or
- \( \beta_6 = 0, 0101\ldots11 \)
  \( (b_m \text{ is the last one}) \)
if $m$ is odd.

Now we check easily that

$$\sum_{u=0}^{m-1} \|2^u \beta_k\|^2 = \frac{m}{9} + \frac{1}{9} - \frac{(-1)^m}{27 \cdot 2^m} - \frac{1}{27 \cdot 2^{2m}}$$

for $k = 1, \ldots, 6$ and the result follows. $\Box$

We give the Proof of Corollary 2.3. We have

$$\max_{N \leq 2^m} \sum_{u=1}^{m} \|N\|_{2u}^2 = \max_{\beta_{m-\text{bit}}} \sum_{u=0}^{m-1} \|2^u \beta\|^2 = \frac{m}{9} + \frac{1}{9} - \frac{(-1)^m}{27 \cdot 2^m} - \frac{1}{27 \cdot 2^{2m}}$$

by Lemma 5.2. The result follows now together with (2). $\Box$

For the proof of Corollary 3.2 we can in principle proceed as for the proof of Corollary 2.3. However, in this case the detailed computations are by far more involved than above. First we have

**Lemma 5.3.** Assume that $\beta = 0, b_1 b_2 \ldots$ has two equal consecutive digits $b_i b_{i+1}$ with $i \leq m - 1$ and let $i$ be minimal with this property, i.e.,

- $\beta = 0, 01 \ldots 0100b_{i+2} \ldots$ or
- $\beta = 0, 10 \ldots 100b_{i+2} \ldots$ or
- $\beta = 0, 01 \ldots 1011b_{i+2} \ldots$ or
- $\beta = 0, 10 \ldots 1011b_{i+2} \ldots$.

Replace $\beta$ by

- $\gamma = 0, 01 \ldots 1010b_{i+2} \ldots$ resp.
- $\gamma = 0, 01 \ldots 1010b_{i+2} \ldots$ resp.
- $\gamma = 0, 10 \ldots 0101b_{i+2} \ldots$ resp.
- $\gamma = 0, 01 \ldots 0101b_{i+2} \ldots$.

Then

$$\sum_{u=0}^{m-1} ||2^u \gamma||^2 (m - u) = \sum_{u=0}^{m-1} ||2^u \beta||^2 (m - u) +$$

$$\begin{cases} \left\{ \frac{m}{9} \left( 1 - \frac{(-1)^i}{2^i} \right)^2 - \frac{i}{9} + \frac{4}{27 \cdot 2^i} \left( \frac{1}{2^i} - (-1)^i \right) \right\} (1 - \tau) in the first two cases, \\
\left\{ \frac{m}{9} \left( 1 - \frac{(-1)^i}{2^i} \right)^2 - \frac{i}{9} + \frac{4}{27 \cdot 2^i} \left( \frac{1}{2^i} - (-1)^i \right) \right\} \tau in the last two cases, \\
\end{cases}$$

where $\tau := 0, b_{i+2} b_{i+3} \ldots$.

**Remark.** In any case, for $m > 3$, we have $\sum_{u=0}^{m-1} ||2^u \gamma||^2 (m - u) \geq \sum_{u=0}^{m-1} ||2^u \beta||^2 (m - u)$. 
Proof of Lemma 5.3. We have
\[ m \sum_{u=0}^{m-1} (\|2^u \gamma\|^2 - \|2^u \beta\|^2) (m - u) \]
\[ = m \|\gamma\|^2 - m (m - i) \|2^i \beta\|^2 + \sum_{u=0}^{m-1} (\|2^u (2^i \gamma)\|^2 (m - u - 1) - \|2^u \beta\|^2 (m - u)) . \]
The result now follows as in the proof of Lemma 5.1 by some tedious but straightforward algebra. □

With Lemma 5.3 we obtain

Lemma 5.4. We have
\[ \max_{\beta \text{ m-bit}} \sum_{u=0}^{m-1} \|2^u \beta\|^2 (m - u) \]
\[ = \begin{cases} \frac{m^2}{18} + \frac{m}{18} + \frac{8}{81} + \frac{1}{2^m} \left( \frac{27}{27} \left( 1 - \frac{1}{2^m} \right) (m + \frac{3}{2}) - \frac{8}{81 2^{2m}} \right) & \text{if } m \text{ is even}, \\ \frac{m^2}{18} + \frac{m}{18} + \frac{8}{81} + \frac{1}{2^m} \left( \frac{27}{27} + \frac{1}{27} (1 - \frac{1}{2^m}) (m + \frac{3}{2}) \right) & \text{if } m \text{ is odd}. \end{cases} \]

For even \( m \) the maximum is attained if and only if
\[ \beta = \begin{cases} 0, 0101 \ldots 0110 = \frac{1}{3} \left( 1 + \frac{1}{2^m} \right) & \text{and} \\ 0, 1010 \ldots 1010 = \frac{2}{3} \left( 1 - \frac{1}{2^m} \right) . \end{cases} \]

For odd \( m \) the maximum is attained if and only if
\[ \beta = \begin{cases} 0, 0101 \ldots 011 = \frac{1}{3} \left( 1 + \frac{1}{2^m} \right) & \text{and} \\ 0, 1010 \ldots 0101 = \frac{2}{3} \left( 1 - \frac{1}{2^m} \right) . \end{cases} \]

Proof. For short we write \( f_m(\beta) := \sum_{u=0}^{m-1} \|2^u \beta\|^2 (m - u) \). Let \( m \geq 2 \) be even. It follows from Lemma 5.3 that the \( m \)-bit number \( \beta \) which maximizes our sum has to be of the form
\[ \beta_1 = 0, 0101 \ldots 01 b_{m-1} b_m \quad \text{or} \quad \beta_2 = 0, 1010 \ldots 0 b_{m-1} b_m . \]

First we deal with \( \beta_1 = 0, 0101 \ldots 01 b_{m-1} b_m \). Now we consider four cases corresponding to the possible choices for \( b_{m-1} \) and \( b_m \).

- If \( (b_{m-1}, b_m) = (0, 0) \), then
\[ f_m(\beta_1) = \frac{m^2}{18} + \frac{m}{18} - \frac{1}{81} - \frac{8}{27 2^m} - \frac{16}{27 2^{2m}} - \frac{16}{81 2^m} - \frac{64}{81 2^{2m}} . \]

- If \( (b_{m-1}, b_m) = (1, 1) \), then
\[ f_m(\beta_1) = \frac{m^2}{18} + \frac{m}{18} - \frac{1}{81} + \frac{10}{27 2^m} - \frac{25}{27 2^{2m}} + \frac{20}{81 2^m} - \frac{100}{81 2^{2m}} . \]

- If \( (b_{m-1}, b_m) = (1, 0) \), then
\[ f_m(\beta_1) = \frac{m^2}{18} + \frac{m}{18} + \frac{8}{27 2^m} - \frac{4}{27 2^{2m}} + \frac{8}{81 2^m} - \frac{16}{81 2^{2m}} . \]
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• If \((b_{m−1}, b_m) = (0, 1)\), then
  
  \[ f_m(β_1) = \frac{m^2}{18} + \frac{m}{18} + \frac{8}{81} - \frac{2}{27} \frac{m}{2^m} - \frac{1}{27} \frac{m}{2^{2m}} - \frac{4}{81} \frac{1}{2^m} - \frac{4}{81} \frac{1}{2^{2m}}. \]

  Therefore we find that the choice \((b_{m−1}b_m) = (1, 0)\) gives the maximal value, i.e., \(β_1 = 0, 0101 \ldots 0110\). For \(β_2 = 0, 1010 \ldots 10\) we find in the same way that \((b_{m−1}, b_m) = (1, 0)\) gives the maximal value, i.e., \(β_2 = 0, 1010 \ldots 1010\). Since
  
  \[ f_m(β_1) = f_m(β_2) = \frac{m^2}{18} + \frac{m}{18} + \frac{8}{81} + \frac{1}{2^m} \left( \frac{4}{27} \left( 1 - \frac{1}{2^m} \right) \left( m + \frac{2}{3} \right) - \frac{8}{81} \cdot 2^m \right) \]

  the result follows for even \(m ≥ 2\).

  For odd \(m ≥ 3\) the result can be proved analogously. □

We give the Proof of Corollary 3.2. We have

\[
\max_{N \leq 2^m} \sum_{u=1}^{m} \| N \|^{2u} = \max_{β = \text{bit}} \sum_{u=0}^{m−1} \| 2^u β \|^2 (m - u) = \frac{m^2}{18} + \frac{m}{18} + \frac{8}{81} + O \left( \frac{m}{2^m} \right)
\]

by Lemma 5.4. The result follows now together with (3). □

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References

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