DOUBLE $n$-ARY RELATIONAL STRUCTURES

Jiří Karásek, Brno

(Received January 22, 1996)

Abstract. In [7], V. Novák and M. Novotný studied ternary relational structures by means of pairs of binary structures; they obtained the so-called double binary structures. In this paper, the idea is generalized to relational structures of any finite arity.

Keywords: $n$-ary relation, $n$-ary structure, binding relation, double $n$-ary structure

MSC 1991: 04A05, 08A02

Let $G$ be a set, let $n \geq 2$ be an integer. As usual, an $n$-ary relation on $G$ is defined as a set $R \subseteq G^n$. The pair $G = (G, R)$ is then called an $n$-ary relational structure (or briefly an $n$-ary structure). An $n$-ary structure $G = (G, R)$ (and the relation $R$ on $G$ as well) is called

*symmetric* if $(x_1, x_2, \ldots, x_n) \in R$ implies $(x_n, x_{n-1}, \ldots, x_1) \in R$ for any $x_1, x_2, \ldots, x_{n-1}, x_n \in G$;

*asymmetric* if $(x_1, x_2, \ldots, x_n) \in R$ implies $(x_n, x_{n-1}, \ldots, x_1) \notin R$ for any $x_1, x_2, \ldots, x_{n-1}, x_n \in G$;

*cyclic* if $(x_1, x_2, \ldots, x_n) \in R$ implies $(x_2, x_3, \ldots, x_n, x_1) \in R$ for any $x_1, x_2, x_3, \ldots, x_n \in G$;

*transitive* if $(x_1, x_2, \ldots, x_n) \in R$, $(x_n, x_{n-1}, \ldots, x_2, x_{n+1}) \in R$ imply $(x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}) \in R$ for any $x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1} \in G$;

*weakly transitive* if $(x, y, y, \ldots, y) \in R$, $(y, y, \ldots, y, z) \in R$ imply $(x, y, y, \ldots, y, z) \in R$ for any $x, y, z \in G$.

For any $\alpha = (x_1, x_2, \ldots, x_n) \in G^n$, put $\alpha^{-1} = (x_n, x_{n-1}, \ldots, x_1)$, $\alpha' = (x_{n-1}, x_n, \ldots, x_1, x_n)$.

Let $\rho$ be an $n$-ary relation on $G$, let $r$ be a binary relation on $\rho$ with the property: If $\alpha = (x_1, x_2, \ldots, x_n) \in \rho$, $\beta = (y_1, y_2, \ldots, y_n) \in \rho$, $(\alpha, \beta) \in r$, then $x_{j+1} = y_j$ for $j = 1, 2, \ldots, n-1$. Then $r$ is called a *binding relation* on $\rho$. 
Let \( \varrho \) be an \( n \)-ary relation on \( G \), let \( r \) be a binding relation on \( \varrho \). Then the triple \( G = (G, \varrho, r) \) is called a double \( n \)-ary relational structure (or briefly a double \( n \)-ary structure). An element \( \alpha \in \varrho \) is called isolated in \( G \) if \((\alpha, \beta) \notin r \) and \((\beta, \alpha) \notin r \) for any \( \beta \in \varrho \). The set of all isolated elements in \( G \) is denoted by \( \varrho_i \).

A double \( n \)-ary structure \( G = (G, \varrho, r) \) (and its binary relation \( r \)) is called

- *inversely symmetric* if \((\alpha, \beta) \in r \) implies \((\beta^{-1}, \alpha^{-1}) \in r \) for any \( \alpha, \beta \in \varrho \);
- *inversely asymmetric* if \((\alpha, \beta) \in r \) implies \((\beta^{-1}, \alpha^{-1}) \notin r \) for any \( \alpha, \beta \in \varrho \);
- *transitive* if \((\alpha, \beta) \in r \) implies the existence of elements \( \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \varrho \) such that \((\beta, \alpha_1) \in r \), \((\alpha_j, \alpha_{j+1}) \in r \) for \( j = 1, 2, \ldots, n-2 \), \((\alpha_{n-1}, \alpha) \in r \) for any \( \alpha, \beta \in \varrho \);
- *reversely transitive* if \((\alpha, \beta) \in r \), \((\beta^{-1}, \gamma) \in r \) imply \((\alpha, \gamma) \in r \) for any \( \alpha, \beta, \gamma \in \varrho \).

Let \( G = (G, \varrho, r) \) be a double \( n \)-ary structure. Define an \((n+1)\)-ary relation \( R \) on \( G \) as follows:

\[ (x_1, x_2, \ldots, x_n, x_{n+1}) \in R \iff (x_1, x_2, \ldots, x_n) = \alpha \in \varrho, (x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varrho, (\alpha, \beta) \in r \] for any \( x_1, x_2, x_3, \ldots, x_n, x_{n+1} \in G \). Denote \( U(G) = (G, R) \). Then \( U(G) \) is an \((n+1)\)-ary structure.

If we denote by \( 2\mathcal{R}_n \) the class of all double \( n \)-ary structures, and by \( \mathcal{R}_{n+1} \) the class of all \((n+1)\)-ary structures, then \( U \) is a map of \( 2\mathcal{R}_n \) into \( \mathcal{R}_{n+1} \).

Now, let \( G = (G, R) \) be an \((n+1)\)-ary structure. Define an \( n \)-ary relation \( \varrho \) on \( G \) as follows:

\[ (x_1, x_2, \ldots, x_n) \in \varrho \iff \text{there exists } t \in G \text{ such that } (x_1, x_2, \ldots, x_n, t) \in R \text{ or } (t, x_1, x_2, \ldots, x_n) \in R \text{ for any } x_1, x_2, \ldots, x_n \in G \]; further, define a binary relation \( r \) on \( \varrho \) as follows:

\[ (\alpha, \beta) \in r \iff (x_1, x_2, \ldots, x_n) \in \varrho, (\alpha, \beta) \in r \] for any \( \alpha, \beta \in \varrho \) with \( (\alpha, \beta) \in r \).

Denote \( L(G) = (G, \varrho, r) \). Then \( L(G) \) is a double \( n \)-ary structure and \( L \) is a map of \( \mathcal{R}_{n+1} \) into \( 2\mathcal{R}_n \).

Moreover, denote by \( 2\mathcal{R}_n^i \) the class of all double \( n \)-ary structures without isolated elements.

1. **Theorem.** Let \( G \) be an \((n+1)\)-ary structure. Then \((U \cdot L)(G) = G\), i.e., \( U \cdot L = \text{id}_{\mathcal{R}_{n+1}} \).

**Proof.** Let \( G = (G, R) \), \( L(G) = (G, \varrho, r) \), \((U \cdot L)(G) = (G, R')\). Let \((x_1, x_2, \ldots, x_n, x_{n+1}) \in R\). By the definition of \( L \), we have \((x_1, x_2, \ldots, x_n) = \alpha \in \varrho, (x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varrho, (\alpha, \beta) \in r \). By the definition of \( U \), we have \((x_1, x_2, \ldots, x_n, x_{n+1}) \in R'\). Thus \( R \subseteq R' \). Let \((x_1, x_2, \ldots, x_n, x_{n+1}) \in R'\). Then, by the definition of \( U \), \((x_1, x_2, \ldots, x_n) = \alpha \in \varrho, (x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varrho, (\alpha, \beta) \in r \). By the definition of \( L \), \((x_1, x_2, \ldots, x_{n+1}) \in R \). Hence \( R' \subseteq R \). Summarizing, we conclude \( R = R' \).
2. Theorem. Let $G = (G, g, r)$ be a double $n$-ary structure and let $(L \cdot U)(G) = (G, g', r')$. Then $g' = g - g_i, r' = r$, i.e. $L \cdot U|_{2 \mathcal{R}_n} = \text{id}_{2 \mathcal{R}_n}$.

Proof. Denote $U(G) = (G, R)$. Let $(x_1, x_2, ..., x_n) \in g'$. Then, by the definition of $L$, there exists $t \in G$ such that $(x_1, x_2, ..., x_n, t) \in R$ or $(t, x_1, x_2, ..., x_n) \in R$. In the first case, by the definition of $U$, we have $(x_1, x_2, ..., x_n) = \alpha \in g$, $(x_2, x_3, ..., x_n, t) = \beta \in g_i \alpha (\alpha, \beta) \in r$, thus the element $\alpha \in g$ is not isolated, so that $\alpha \in g - g_i$. In the second case, $(t, x_1, x_2, ..., x_n, t) = \alpha \in g$, $(x_1, x_2, ..., x_n, t) = \beta \in g_i \alpha (\alpha, \beta) \in r$, hence the element $\beta \in g$ is not isolated and $\beta \in g - g_i$. We have $g' \subseteq g - g_i$. Let, on the contrary, $(x_1, x_2, ..., x_n) \in g - g_i$. Then there exists $\beta \in g$ such that $\alpha, \beta \in r$ or $(\alpha, \beta) \in r$. In the first case we have $\beta = (x_1, x_2, ..., x_n, t)$ for some $t \in G$, therefore, by the definition of $U$, $(x_1, x_2, ..., x_n, t) \in R$ and, by the definition of $L$, $\alpha \in g'$. The second case is analogous. Hence $g - g_i \subseteq g'$. Altogether, we have $g' = g - g_i$.

Let $(\alpha, \beta) \in r'$. By the definition of $L$, $\alpha = (x_1, x_2, ..., x_n), \beta = (x_2, x_3, ..., x_n, x_{n+1}) \in R$ for some $x_1, x_2, x_3, ..., x_n, x_{n+1} \in G$, $(x_1, x_2, ..., x_n, x_{n+1}) \in R$. This implies, by the definition of $U$, $\alpha \in g, \beta \in g, (\alpha, \beta) \in r$. Thus $r' \subseteq r$. Let $(\alpha, \beta) \in r$. Then $\alpha = (x_1, x_2, ..., x_n) \in g, \beta = (x_2, x_3, ..., x_n, x_{n+1}) \in g$ for some $x_1, x_2, x_3, ..., x_n, x_{n+1} \in G$, hence, by the definition of $U$, we have $(x_1, x_2, ..., x_n, x_{n+1}) \in R$. Consequently, by the definition of $L$, $\alpha \in g', \beta \in g', (\alpha, \beta) \in r'$, and $r \subseteq r'$. Summarizing, we obtain $r = r'$.

In the case that $G$ contains no isolated elements, we have $g_i = 0$, thus $g = g'$, $r = r'$, so that $L \cdot U|_{2 \mathcal{R}_n} = \text{id}_{2 \mathcal{R}_n}$.

Denote by $2 \mathcal{R}_n$ the category whose class of objects is $2 \mathcal{R}_n$ and whose morphisms are maps preserving both relations, i.e., for $G = (G, g, r), H = (H, \sigma, s) \in 2 \mathcal{R}_n$, a map $f : G \to H$ is a morphism if $(x_1, x_2, ..., x_n) \in g$ implies $(f(x_1), f(x_2), ..., f(x_n)) \in \sigma$, and $(x_1, x_2, ..., x_n, (x_2, x_3, ..., x_n, x_{n+1})) \in r$ implies $((f(x_1), f(x_2), ..., f(x_n)), (f(x_2), f(x_3), ..., f(x_{n+1}))) \in s$ for any $x_1, x_2, x_3, ..., x_n, x_{n+1} \in G$.

Further, denote by $\mathcal{R}_{n+1}$ the category whose class of objects is $\mathcal{R}_{n+1}$ and whose morphisms are maps preserving the relation, i.e., for $G = (G, H), H = (H, S) \in \mathcal{R}_{n+1}$ a map $f : G \to H$ is a morphism if $(x_1, x_2, ..., x_n, x_{n+1}) \in R$ implies $(f(x_1), f(x_2), ..., f(x_n)) \in S$ for any $x_1, x_2, ..., x_n, x_{n+1} \in G$.

Moreover, for any morphism $f \in \text{Hom}_{\mathcal{R}_n}(G, H)$, where $G = (G, g, r), H = (H, \sigma, s)$, denote $U(f) = f$. Similarly, for any morphism $f \in \text{Hom}_{\mathcal{R}_{n+1}}(G, H)$, denote $L(f) = f$.

3. Theorem. $U$ is a covariant functor from the category $2 \mathcal{R}_n$ to the category $\mathcal{R}_{n+1}$, $L$ is a covariant functor from the category $\mathcal{R}_{n+1}$ to the category $2 \mathcal{R}_n$.

Proof. Let $f \in \text{Hom}_{\mathcal{R}_n}(G, H), H = (G, g, r), G = (G, R), H = (H, \sigma, s), U(H) = (H, S)$. Let $(x_1, x_2, ..., x_n, x_{n+1}) \in R$. Then $(x_1, x_2, ..., x_n) \in G,$
4. **Theorem.** Let $G$ be a double $n$-ary structure. Then the following assertions hold:

(i) $G$ is symmetric if and only if $U(G)$ is symmetric.

(ii) $G$ is asymmetric if and only if $U(G)$ is asymmetric.

**Proof.** Let $G = (G, g, r), U(G) = (G, R)$.

(i) Let $G$ be symmetric and let $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$. Then

\[(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n, x_{n+1})\] in $G$ implies $\alpha = (\beta) \in \sigma$, $(\alpha, \beta) \in r$. This implies $(\beta^{-1}, \alpha^{-1}) \in r$, so $(x_{n+1}, x_n, \ldots, x_2, x_1) \in G$, $\beta = (x_2, x_3, \ldots, x_n, x_{n+1}) \in G$. This implies $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$, so that $(x_{n+1}, x_n, \ldots, x_2, x_1) \in R$, i.e. $(x_{n+1}, x_n, \ldots, x_2, x_1) = (x_{n+1}, x_n, \ldots, x_2, x_1) \in G$, hence $(\beta^{-1}, \alpha^{-1}) \in r$ and $G$ is symmetric.

(ii) Let $G$ be asymmetric and let $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R$. Then again $(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n, x_{n+1}) \in G$ implies $(\beta^{-1}, \alpha^{-1}) \notin r$. But $\beta^{-1} = (x_2, x_3, \ldots, x_n, x_{n+1}) \notin R$, and $U(G)$ is asymmetric. Let $U(G)$ be asymmetric and let $(\alpha, \beta) \in r$. Then there exist elements $x_1, x_2, \ldots, x_n, x_{n+1} \in G$ such that $(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n, x_{n+1}) \notin R$. Consequently $(x_{n+1}, x_n, \ldots, x_2, x_1) = (x_{n+1}, x_n, \ldots, x_2, x_1) \notin R$. Hence, we have $(\beta^{-1}, \alpha^{-1}) \notin r$, and $G$ is asymmetric.

5. **Theorem.** Let $G$ be an $(n+1)$-ary structure. Then the following assertions hold:

(i) $G$ is symmetric if and only if $L(G)$ is inversely symmetric.

(ii) $G$ is asymmetric if and only if $L(G)$ is inversely asymmetric.

**Proof.** (i) If $L(G)$ is inversely symmetric, then, by 4, $U(L(G))$ is symmetric. But, by 1, $U(L(G)) = G$. If $G = U(L(G))$ is symmetric, then, by 4, $L(G)$ is inversely symmetric.
(ii) If \( L(G) \) is inversely asymmetric, then, by 4, \( U(L(G)) \) is asymmetric. But \( U(L(G)) = G \). If \( G = U(L(G)) \) is asymmetric, then, by 4, \( L(G) \) is inversely asymmetric.

\[ \square \]

6. Theorem. Let \( G \) be a double n-ary structure. Then \( G \) is transferable if and only if \( U(G) \) is cyclic.

Proof. Let \( G = (G, \varnothing, r) \), \( U(G) = (G, R) \). Let \( G \) be transferable and let \( (x_1, x_2, \ldots, x_n, x_{n+1}) \in R \). Then \((x_1, x_2, \ldots, x_n) = \alpha \in \varnothing, (x_2, x_3, \ldots, x_n, x_{n+1}) = \beta \in \varnothing, (\alpha, \beta) \in r \). Thus, there exist \( \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \varnothing \) such that \((\beta, \alpha_1) \in r, (\alpha_2, \alpha_{j+1}) \in r \) for \( j = 1, 2, \ldots, n-2 \) and \((\alpha_{n-1}, \alpha) \in r \). Denote \( \alpha_0 = \beta, \alpha_n = \alpha \). Then we have \((\alpha_j, \alpha_{j+1}) \) for \( j = 0, 1, 2, \ldots, n-1 \). We shall show by induction that \( \alpha_j = (x_{j+2}, x_{j+3}, \ldots, x_n, x_{n+1}, x_1, x_2, \ldots, x_j) \) for \( j = 0, 1, 2, \ldots, n \). For \( j = 0 \) it is true. Let \( 0 < j_0 \leq n \). Let the preceding hold for each \( j, 0 \leq j < j_0 \). As \((\alpha_{j_0-1}, \alpha_{j_0}) \in r \) and \( r \) is binding, there exists \( y \in G \) such that \( \alpha_{j_0} = (x_{j_0+2}, x_{j_0+3}, \ldots, x_1, \ldots, x_{j_0-1}, y) \). We shall show by another induction that \( \alpha_{j+k} \) has \( y \) on the \((n-k)\)-th position, for \( k = 0, 1, 2, \ldots, n-j_0 \). For \( k = 0 \) it is true. Let \( 0 < k_0 \leq n-j_0 \). As \((\alpha_{j+k_0-1}, \alpha_{j+k_0}) \in r, \alpha_{j+k_0} \) has \( y \) on the \((n-k_0)\)-th position, \( r \) is binding, \( \alpha_{j+k_0} \) has \( y \) on the \((n-k_0)\)-th position. Particularly, \( \alpha_n \) has \( y \) on the \( j_0\)-th position, hence \( y = x_{j_0} \). Thus, we have \( \beta = (x_2, x_3, \ldots, x_n, x_{n+1}) \in \varnothing, \alpha_1 = (x_3, x_4, \ldots, x_n, x_{n+1}, x_1) \in \varnothing, (\beta, \alpha_1) \in r, \) so that \( (x_2, x_3, \ldots, x_n, x_{n+1}, x_1) \in R \) and \( U(G) \) is cyclic.

Let, on the contrary, \( U(G) \) be cyclic and let \((\alpha, \beta) \in r \). Then there exist elements \( x_1, x_2, \ldots, x_n, x_{n+1} \in G \) such that \( \alpha = (x_1, x_2, \ldots, x_n) \in \varnothing, \beta = (x_2, x_3, \ldots, x_n, x_{n+1}) \in \varnothing \). Thus \((x_1, x_2, \ldots, x_n, x_{n+1}) \in R \). Hence \((x_2, x_3, \ldots, x_n, x_{n+1}, x_1) \in R, (x_3, x_4, \ldots, x_n, x_{n+1}, x_1, x_2) \in R, \ldots, (x_{n+1}, x_1, x_2, \ldots, x_n) \in R \). Denote \( \alpha_1 = (x_3, x_4, \ldots, x_n, x_{n+1}, x_1), \alpha_2 = (x_4, x_5, \ldots, x_n, x_{n+1}, x_1, x_2), \ldots, \alpha_{n-1} = (x_{n+1}, x_1, x_2, \ldots, x_{n-1}) \). Then \( \alpha_j \in \varnothing \) for \( j = 1, 2, \ldots, n-1 \). \( (\alpha, \beta) \in r, (\alpha_j, \alpha_{j+1}) \in r \) for \( j = 1, 2, \ldots, n-2 \). Consequently, \( G \) is transferable.

\[ \square \]

7. Theorem. Let \( L(G) \) be an \((n+1)\)-ary structure. Then \( G \) is cyclic if and only if \( L(G) \) is transferable.

Proof. Let \( L(G) \) be transferable. By 6, \( U(L(G)) \) is cyclic. But, by 1, \( G = U(L(G)) \).

Let \( G = U(L(G)) \) be cyclic. Then, by 6, \( L(G) \) is transferable.

\[ \square \]

8. Theorem. Let \( G = (G, \varnothing, r) \) be a double n-ary structure. If the binary relation \( r \) is transitive, then \( U(G) \) is weakly transitive.

Proof. Let \( U(G) = (G, R) \) and let \((x, y, y, \ldots, y) \in R, (y, y, \ldots, y, z) \in R \). Then \( \alpha = (x, y, y, \ldots, y) \in \varnothing, \beta = (y, y, \ldots, y) \in \varnothing, \gamma = (y, y, \ldots, y, z) \in \varnothing, (\alpha, \beta) \in \varnothing \).
r, \((\beta, \gamma) \in r\). Hence \((\alpha, \gamma) \in r\), so that \((x, y, \ldots, y, z) \in R\) and \(U(G)\) is weakly transitive.

9. Remark. The converse of 8 does not hold, which can be easily shown by a counterexample.

10. Theorem. Let \(G\) be a double \(n\)-ary structure. Then \(G\) is reversely transitive if and only if \(U(G)\) is transitive.

Proof. Let \(G = (G, \theta, r)\), \(U(G) = (G, R)\). Let \(G\) be reversely transitive, let \((x_1, x_2, \ldots, x_n, x_{n+1}) \in R\), \((x_{n+1}, x_n, \ldots, x_2, x_1) \in R\). Then, by the definition of \(U\), \((x_1, x_2, \ldots, x_n) = \alpha \in g\), \((x_2, x_3, \ldots, x_{n+1}) = \beta \in g\), \((\alpha, \beta) \in r\), \((x_{n+1}, x_n, \ldots, x_2) = \beta^{-1} \in g\), \((x_n, x_{n-1}, \ldots, x_2, x_{n+2}) = \gamma' \in g\). As \(G\) is reversely transitive, we have \((\alpha, \gamma) \in r\). But \(\gamma = (x_2, x_3, \ldots, x_n, x_{n+2}) \in g\), hence \((x_1, x_2, \ldots, x_n, x_{n+2}) \in R\) and \(U(G)\) is transitive.

Let \(U(G)\) be transitive and let \(\alpha, \beta, \gamma \in g\), \((\alpha, \beta) \in r\), \((\beta^{-1}, \gamma') \in r\). There exist elements \(x_1, x_2, \ldots, x_n, x_{n+2} \in G\) such that \(\alpha = (x_1, x_2, \ldots, x_n)\), \(\beta = (x_2, x_3, \ldots, x_n, x_{n+1})\) (for \(r\) is binding), \(\gamma = (x_2, x_3, \ldots, x_n, x_{n+2})\) (for \(\beta^{-1} = (x_{n+1}, x_n, \ldots, x_2, x_1)\)), \(\gamma' = (x_n, x_{n-1}, \ldots, x_2, x_1)\) and \(r\) is binding. Hence \((x_1, x_2, \ldots, x_n, x_{n+1}) \in R\), \((x_{n+1}, x_n, \ldots, x_2, x_1) \in R\), so that \((x_1, x_2, \ldots, x_n, x_{n+2}) \in R\) for \(U(G)\) is transitive. Consequently, \((\alpha, \gamma) \in r\) and \(G\) is reversely transitive.

11. Theorem. Let \(G\) be an \((n+1)\)-ary structure. Then \(G\) is transitive if and only if \(L(G)\) is reversely transitive.

Proof. By 1, \(U(L(G)) = G\). Hence \(L(G)\) is reversely transitive if and only if \(U(L(G)) = G\) is transitive, by 10.

\[
\text{References}
\]


Author's address: Jiří Karásek, Technical University, Technická 2, 61669 Brno, Czech Republic.

174