CLOSED CONVEX $\ell$-SUBGROUPS AND RADICAL CLASSES OF CONVERGENCE $\ell$-GROUPS

Ján Jakubík, Košice

(Received March 20, 1996)

Abstract. In this paper we prove that the system of all closed convex $\ell$-subgroups of a convergence $\ell$-group is a Brouwer lattice and that a similar result is valid for radical classes of convergence $\ell$-groups.

Keywords: convergence $\ell$-group, closed convex $\ell$-subgroup, radical class of convergence $\ell$-groups

MSC 1991: 06F20, 22C05

All $\ell$-groups considered in the present paper are assumed to be abelian. For convergence $\ell$-groups (shorter: cl-groups) we apply the same notation and definitions as in [6].

Let $(G, \alpha)$ be a cl-group (where $G$ is an $\ell$-group and $\alpha$ is a convergence in $G$). For the definition of a closed $\ell$-subgroup (shorter: cl-subgroup) of $(G, \alpha)$ cf. Section 1 below. The system of all convex cl-subgroups of $(G, \alpha)$ will be denoted by $c(G, \alpha)$; this system is partially ordered by the set-theoretical inclusion.

In the present paper we prove that $c(G, \alpha)$ is a Brouwer lattice. The lattice operations in $c(G, \alpha)$ are constructively described.

For $X \subseteq G$ the meaning of $\lim X$ is defined in a natural way. We show that if $X$ is an $\ell$-subgroup of $G$ such that $X$ can be represented as a direct product of a finite number of linearly ordered groups, then

$$\lim_\alpha \lim_\alpha X = \lim_\alpha X.$$

A nonempty class $A$ of cl-groups is called a radical class of cl-groups if it is closed with respect to isomorphisms, convex cl-subgroups and joins of convex cl-subgroups. For radical classes $A_1$ and $A_2$ we put $A_1 \subseteq A_2$ if $A_1$ is a subclass of $A_2$.  

301
We prove that a certain form of distributive law (analogous to the condition applied when defining a Brouwer lattice) is valid for radical classes of cl-groups.

The analogous notion of a radical class of \( \ell \)-groups was introduced in [5] and studied in several papers (cf., e.g., [1], [2], [7], [8]).

1. Preliminaries

For an \( \ell \)-group \( G \) we denote by \( c(G) \) the system of all convex \( \ell \)-subgroups of \( G \); this system is partially ordered by the set theoretical inclusion. Then \( c(G) \) is a complete lattice. The lattice operations in \( c(G) \) will be denoted by \( \lor \) and \( \land \).

Let \( \mathcal{G} \) be the class of all \( \ell \)-groups. A nonempty subclass \( X \) of \( \mathcal{G} \) is said to be a radical class of \( \ell \)-groups if it satisfies the following conditions:

(i) \( X \) is closed with respect to isomorphisms.
(ii) Whenever \( G \in X \) and \( G_1 \in c(G) \), then \( G_1 \in X \).
(iii) Whenever \( G \in \mathcal{G} \) and \( \{G_i\}_{i \in I} \) is a nonempty subset of \( X \cap c(G) \), then \( \bigvee_{i \in I} G_i \) belongs to \( X \).

We suppose that the reader is acquainted with the definitions from Section 1 of [6].

Let \( (G, \alpha) \) and \( (G_1, \alpha_1) \) be cl-groups.

1.1. Definition. \( (G_1, \alpha_1) \) is said to be a cl-subgroup of \( (G, \alpha) \) if

(i) \( G_1 \) is an \( \ell \)-subgroup of \( G \);
(ii) whenever \( (x_n) \) is a sequence in \( G_1, x \in G \) and \( x_n \to_\alpha x \), then \( x \in G_1 \) and \( x_n \to_{\alpha_1} x \);
(iii) whenever \( (x_n) \) is a sequence in \( G_1, x \in G_1 \) and \( x_n \to_{\alpha_1} x \), then \( x_n \to_\alpha x \).

If \( (G_1, \alpha_1) \) is a cl-subgroup of \( (G, \alpha) \), then we often write \( (G_1, \alpha) \) instead of \( (G_1, \alpha_1) \).

The meaning of a convex cl-subgroup of \( (G, \alpha) \) is obvious. The system of all convex cl-subgroups of \( (G, \alpha) \) will be denoted by \( c(G, \alpha) \). If \( (G_1, \alpha_1) \) and \( (G_2, \alpha_2) \) belong to \( c(G, \alpha) \) and \( G_1 \subseteq G_2 \), then we put \( (G_1, \alpha_1) \leq (G_2, \alpha_2) \). It is easy to verify that under the relation \( \leq \), the system \( c(G, \alpha) \) is a complete lattice. The lattice operations in \( c(G, \alpha) \) will be denoted by \( \lor^c \) and \( \land^c \).

1.2. Definition. A mapping \( \varphi \) of \( G \) into \( G_1 \) is called a cl-homomorphism if

(i) \( \varphi \) is a homomorphism of the \( \ell \)-group \( G \) into the \( \ell \)-group \( G_1 \);
(ii) whenever \( (x_n) \) is a sequence in \( G, x \in G \) and \( x_n \to_\alpha x \), then \( \varphi(x_n) \to_{\alpha_1} \varphi(x) \).

If there exists a cl-homomorphism of \( (G, \alpha) \) onto \( (G_1, \alpha_1) \), then \( (G_1, \alpha_1) \) is said to be a homomorphic image of \( (G, \alpha) \).
1.3. Definition. Let $\varphi$ be a cl-homomorphism of $(G, \alpha)$ onto $(G_1, \alpha_1)$ such that

(i) $\varphi$ is a monomorphism;
(ii) the inverse mapping $\varphi^{-1}$ is a cl-homomorphism of $(G_1, \alpha_1)$ onto $(G, \alpha)$.

Then $\varphi$ is an isomorphism of $(G, \alpha)$ onto $(G_1, \alpha_1)$; if such $\varphi$ does exist, then $(G_1, \alpha_1)$ is said to be cl-isomorphic to $(G, \alpha)$.

Let $\mathcal{G}_c$ be the class of all cl-groups.

1.4. Definition. A nonempty subclass $Y$ of $\mathcal{G}_c$ is said to be a radical class of a cl-group if the following conditions are satisfied:

(i) $Y$ is closed with respect to cl-isomorphisms;
(ii) whenever $(G, \alpha) \in Y$ and $(G_1, \alpha_1) \in c(G, \alpha)$, then $(G_1, \alpha_1) \in Y$;
(iii) whenever $(G, \alpha) \in \mathcal{G}_c$ and $\{(G_i, \alpha_i)\}_{i \in I}$ is a nonempty subset of $Y \cap c(G, \alpha)$, then $\bigvee_{i \in I} (G_i, \alpha_i) \in Y$.

We shall often apply without quotation the following facts:

(a1) If $a_n \rightarrow_\alpha a$ and $a_n \leq a$ for each $n \in \mathbb{N}$, then $\bigvee_{n \in \mathbb{N}} a_n = a$ (and dually).

(a2) If $G$ is linearly ordered, $a_n \rightarrow_\alpha a, c_1 < a < c_2$, then there is $m \in \mathbb{N}$ such that for each $n > m$ the relation $c_1 < a_n < c_2$ is valid.

(The assertion (a1) is easy to verify; (a2) is a consequence of (a1).)

2. The system $c(G, \alpha)$

Again, let $(G, \alpha) \in \mathcal{G}_c$.

A subset $S$ of $G$ is said to be closed with respect to $(G, \alpha)$ if, whenever $(x_n)$ is a sequence in $S, x \in G$ and $x_n \rightarrow_\alpha x$, then $x \in S$.

2.1. Lemma. Let $H$ be an $t$-subgroup of $G$ such that it is closed with respect to $(G, \alpha)$. For a sequence $(x_n)$ in $H$ and $x \in H$ we put $x_n \rightarrow_{\alpha(H)} x$ if $x_n \rightarrow_\alpha x$.

Then

(i) $(H, \alpha(H))$ is a cl-group,
(ii) $(H, \alpha(H))$ is a cl-subgroup of $(G, \alpha)$.

Proof. The first assertion is an immediate consequence of the definition of the cl-group. Since $H$ is closed with respect to $(G, \alpha)$ and in view of (i), the assertion (ii) holds as well. \[\square\]
In view of the above remark concerning the notation (cf. Section 1) we will write \((H, \alpha)\) instead of \((H, \alpha(H))\).

Let \(X\) be a nonempty subset of \(G\). We denote by \(\lim_\alpha X\) the set of all \(y \in G\) such that there exists a sequence \((x_n)\) in \(X\) with \(x_n \to_\alpha y\).

**2.2. Lemma.** Let \(H\) be an \(\ell\)-subgroup of \(G\). Then \(\lim_\alpha H\) is an \(\ell\)-subgroup of \(G\). If, moreover, \(H\) is convex in \(G\), then \(\lim_\alpha H\) is convex in \(\bar{G}\) as well.

**Proof.** Let \(y_1, y_2 \in \lim_\alpha H\). Hence there are sequences \((x_n^i)\) in \(H\) such that \(x_n^1 \to_\alpha y_1\) and \(x_n^2 \to_\alpha y_2\) (\(i = 1, 2\)). Thus \(x_n^1 + x_n^2 \to_\alpha y_1 + y_2\), and analogously for the operations \(\land\) and \(\lor\). Also, \(\neg x_n^1 \to_\alpha \neg y_1\). Hence \(\lim_\alpha H\) is an \(\ell\)-subgroup of \(G\).

Now suppose that \(H\) is convex in \(\bar{G}\) and that \(z \in G\), \(y_1 \leq z \leq y_2\). Then
\[
x_n^1 \land x_n^2 \to_\alpha y_1, \quad x_n^1 \lor x_n^2 \to_\alpha y_2.
\]

Put
\[
z_n = ((x_n^1 \land x_n^2) \lor z) \land (x_n^1 \lor x_n^2).
\]

Hence \(z_n \in H\) and \(z_n \to_\alpha \left((y_1 \lor z) \land y_2 \right) = z\). Thus \(z \in \lim_\alpha H\). \(\square\)

Let \(H\) be as in 2.2. We put \(H_0 = H\) and for each ordinal \(t > 0\) we construct \(H_t\) by transfinite induction as follows. Suppose that for \(t_1 < t\) all \(H_{t_1}\) are already defined and that they are \(\ell\)-subgroups of \(G\) such that, whenever \(t_1 < t_2 < t\), then \(H_{t_1} \subseteq H_{t_2}\). If \(t\) is a limit ordinal, then we put
\[
H_t = \bigcup_{t_1 < t} H_{t_1}.
\]

If \(t\) is non-limit, then there exists \(t_1\) with \(t = t_1 + 1\). In this case we set
\[
H_t = \lim_\alpha H_{t_1}.
\]

There exists an ordinal \(t\) such that \(H_t = H_{t_2}\) whenever \(t_2 > t\). We denote
\[
\lim_\alpha H = H_t.
\]

From 2.1, 2.2 and from the construction of \(\lim_\alpha H\) we immediately obtain

**2.3. Lemma.** Let \(H\) be an \(\ell\)-subgroup of \(G\). Put \(\lim_\alpha H = H^*\). Then
(i) \((H^*, \alpha)\) is a cl-subgroup of \((G, \alpha)\);
(ii) if \((K, \alpha)\) is a cl-subgroup of \((G, \alpha)\) and \(H \subseteq K\), then \(H^* \subseteq K^*\);
(iii) if, moreover, \(H\) is convex in \(G\), then \(H^*\) is convex in \(G\) as well.
2.4. Lemma. Let \( \{ (H_i, \alpha) \}_{i \in I} \) be a nonempty subset of \( c(G, \alpha) \). Put \( H_0 = \bigcap_{i \in I} H_i, \) \( H^0 = \bigvee_{i \in I} H_i \). Then

\[
\begin{align*}
(i) & \quad \bigwedge_{i \in I} (H_i, \alpha) = (H_0, \alpha); \\
(ii) & \quad \bigvee_{i \in I} (H_i, \alpha) = (\lim H^0, \alpha).
\end{align*}
\]

Proof. The first assertion is obvious; the second is a consequence of 2.3. \( \square \)

2.5. Lemma. Let \( H \) be an \( \ell \)-subgroup of \( G \). Then the following conditions are equivalent:

(i) \( H \) is closed with respect to \( (G, \alpha) \);

(ii) \( H^+ \) is closed with respect to \( (G, \alpha) \).

Proof. Let (i) be valid and let \( (x_n) \) be a sequence in \( H^+, x \in G, x_n \to_\alpha x \). Then \( x_n = x_n \vee 0 \to_\alpha x \vee 0 \), whence \( x \vee 0 = x \) and thus (ii) holds. Conversely, suppose that (ii) is satisfied. Let \( (x_n) \) be a sequence in \( H, x \in G, x_n \to_\alpha x \). Then \( x_n^+ \to_\alpha x^+ \) and \( x_n^- \to_\alpha x^- \). We have \( x_n^+, x_n^- \in H^+ \) for each \( n \in \mathbb{N} \) and thus, in view of (ii), both \( x^+ \) and \( x^- \) belong to \( H^+ \). Hence \( x = x^+ - x^- \) is an element of \( H \). \( \square \)

For subsets \( X \) and \( Y \) of \( G \) we denote

\[ X - Y = \{ x - y : x \in X \text{ and } y \in Y \}. \]

2.6. Lemma. Let \( X \) be a subset of \( G^+ \) such that

(i) \( X \) is a sublattice and a subsemigroup of \( G^+ \);
(ii) \( 0 \in X \).

Then \( X - X \) is an \( \ell \)-subgroup of \( G \) and \( (X - X)^+ = X \). If, moreover, \( X \) is a convex subset of \( G^+ \), then \( X - X \) is a convex \( \ell \)-subgroup of \( G \).

The proof is routine, it will be omitted.

For each nonempty subset \( X \) of \( G \) we can perform an analogous construction as we did above for \( H \); in this way we obtain a subset of \( G \) which will be denoted by \( \lim X \) or by \( X^* \).

From the construction of \( X^* \) we immediately obtain

2.7. Lemma. Let \( X \) be as in 2.6. Then

(i) \( X^* \) is a subset of \( G^+ \) and it satisfies the conditions (i), (ii) from 2.6;
(ii) \( X^* \) is closed with respect to \( (G, \alpha) \);
(iii) if, moreover, \( X \) is convex in \( G \), then \( X^* \) is convex in \( G \) as well.

305
2.8. Lemma. Let \( H \) be an \( \ell \)-subgroup of \( G \); put \( X = H^+ \). Then \( H^* = X^* - X^* \).

Proof. In view of the constructions of \( H^* \) and \( X^* \) we have \( X^* \subseteq H^* \). Then according to 2.3 (i), \( X^* - X^* \subseteq H^* \). Further, 2.7 and 2.5 yield that \( X^* - X^* \) is closed with respect to \((G, \alpha)\). Moreover, \( H = H^+ - H^+ \subseteq X^* - X^* \). Hence according to 2.3 (ii) we obtain the relation \( H^* \subseteq X^* - X^* \), which completes the proof. \( \square \)

2.9. Lemma. Let \( \{(H_i, \alpha)\}_{i \in I} \) and \( H^0 \) be as in 2.4. Put \( (H^0)^+ = X \). Then

\[
\bigvee_{i \in I} (H_i, \alpha) = X^* - X^*.
\]

Proof. This is a consequence of 2.4 and 2.8. \( \square \)

Now, let \((A, \alpha)\) and \((B_i, \alpha)\) \((i \in I)\) be elements of \( c(G, \alpha) \). Put

\[
X = X_0 = \left( \bigvee_{i \in I} B_i \right)^+,
\]

and let \( X^* \) be as above. For each ordinal \( t \) we define \( X_t \) analogously as when defining \( H_t \).

Further, we put

\[
Y = Y_0 = \left( \bigvee_{i \in I} (A \wedge B_i) \right)^+;
\]

the symbols \( Y^* \) and \( Y_t \) are defined analogously as \( X^* \) and \( X_t \).

It is well-known that the relation

\[
A \wedge \left( \bigvee_{i \in I} B_i \right) = \bigvee_{i \in I} (A \wedge B_i)
\]

is valid (cf., e.g., [5]). From this relation we immediately obtain that

\[
A \wedge X_0 = Y_0
\]

holds. Let \( t \) be an ordinal with \( t > 0 \) and assume that for each ordinal \( t_1 < t \) the relation

\[
A \wedge X_{t_1} = Y_{t_1}
\]

is valid.

306
a) Suppose that \( t \) is a limit ordinal. Then we have
\[
Y_t = \bigcup_{t_1 < t} Y_{t_1} = \bigcup_{t_1 < t} (A \cap X_{t_1}) = A \cap \left( \bigcup_{t_1 < t} X_{t_1} \right) = A \cap X_t = A \cap X_t.
\]

b) Further, suppose that \( t \) is a non-limit ordinal. Hence there is an ordinal \( t_1 \) with \( t = t_1 + 1 \). Then
\[
X_t = \lim_\alpha X_{t_1}, \quad Y_t = \lim_\alpha Y_{t_1} = \lim_\alpha (A \cap X_{t_1}).
\]

Let \( z \in A \cap X_t \). Hence \( z \in A \) and \( z \in X_t \). Also, \( z \geq 0 \). There exists a sequence \((z_n)\) in \( X_{t_1} \) such that \( z_n \to \alpha z \). Clearly \( z_n \geq 0 \). Then \( 0 \leq z_n \land z \leq z \), whence \( z_n \land z \in A \cap X_{t_1} = Y_{t_1} \) and \( z_n \land z \to \alpha z \). Thus \( z \in Y_{t_1} \) and therefore \( A \cap X_{t_1} \subseteq Y_{t_1} \).

Assume that \( v \in Y_{t_1} \). There exists a sequence \((v_n)\) in \( Y_{t_1} \) with \( v_n \to \alpha v \). We have \( v_n \in A \) for each \( n \in \mathbb{N} \). Since \( A \) is closed with respect to \((G, \alpha)\) we obtain that \( v \in A \). Further, \( v_n \in X_{t_1} \) for each \( n \in \mathbb{N} \) and thus \( v \in X_{t_1} \). Therefore \( v \in A \cap X_t \).

By summarizing, we obtain the relation
\[
A \cap X_t = Y_t
\]
for each ordinal \( t \). Thus
\[
(\ast)
\]
\[
A \cap X^* = Y^*.
\]

2.10. **Theorem.** Let \((A, \alpha)\) and \((B_i, \alpha)\), \( i \in I \), be elements of \((G, \alpha)\). Then
\[
(A, \alpha) \land^c \left( \bigvee_{i \in I} (B_i, \alpha) \right) = \bigvee_{i \in I} (A, \alpha) \land^c (B_i, \alpha)).
\]

**Proof.** This is a consequence of 2.8, 2.9 and of the relation (\(\ast\)).  \(\square\)

2.11. **Corollary.** The system \( c(G, \alpha) \) is a Brouwer lattice.

Let the symbol \( \omega_1 \) have the usual meaning. It is easy to verify that if \( X \) is a nonempty subset of \( G \) and if \( t \) is an ordinal with \( X^* = X_1 \), then \( t \leq \omega_1 \).

If \( t \) is the first ordinal with \( X^* = X_1 \), then \( t \) will be said to be the degree of \( X \) in \((G, \alpha)\).

Further, let \( t' \) be the first ordinal such that, whenever \( X \) is a nonempty subset of \( G \), then the degree of \( X \) in \((G, \alpha)\) is less or equal to \( t' \). We denote \( d(G, \alpha) = t' \).
The following questions remain open:
a) For which ordinals $t$ there exist $(G, \alpha) \in \mathcal{G}_c$ and $X \subseteq G$ such that $t$ is the degree of $X$ in $(G, \alpha)$?
b) For which ordinals $t$ there exists $(G, \alpha) \in \mathcal{G}_c$ such that $d(G, \alpha) = t$?

For a related open question concerning convergence groups cf. [3].

3. The condition \( \lim_{\alpha}^2 X = \lim_{\alpha} X \)

Let $(G, \alpha)$ be as above. For $X \subseteq G$ we denote $\lim_{\alpha}^2 X = \lim_{\alpha} X$. In this section we prove that if $X$ is an $\ell$-subgroup of $G$ such that $X$ is a direct product of a finite number of linearly ordered groups, then the relation

\[
\lim_{\alpha}^2 X = \lim_{\alpha} X
\]

is valid. In other words, the degree of $X$ is either 0 or 1.

3.1. Lemma. Let $X$ be a linearly ordered $\ell$-subgroup of $G$ and $g \in \lim_{\alpha} X$. Then the set $X \cup \{g\}$ is linearly ordered and there are $x^1, x^2 \in X$ such that $x^1 \leq g \leq x^2$.

Proof. In the case $X = \{0\}$ we have $g = 0$. Assume that $X \neq \{0\}$. Then there exists $x_0 \in X$ with $x_0 > 0$. First we prove that the element $g$ cannot be an upper bound of the set $X$. By way of contradiction, suppose that $g > x$ for each $x \in X$.

Since $g \in \lim_{\alpha} X$, there is a sequence $(x_n)$ in $X$ such that $x_n \to_{\alpha} g$. Because $x_n \leq g$ for each $n \in \mathbb{N}$, we obtain that

\[
\sup\{x_n\}_{n \in \mathbb{N}} = g
\]

and this yields that $\sup X = g$. For each $x \in X$ we have $x + x_0 \in X$, thus $x + x_0 \leq g$, hence $x \leq g - x_0 < g$. This is a contradiction with the relation $\sup X = g$. Hence there is $x^2 \in X$ such that $x^2 \nleq g$.

If $x^2$ is any element of $X$ with this property, then there is a positive integer $m(x^2)$ such that for each $n \in \mathbb{N}$ with $n \geq m(x^2)$ we have $x_n \leq x^2$ (otherwise the relation $g \geq x^2$ would be valid). Then $g \geq x^2$. By a dual argument we prove that there is $x^1 \in X$ with $x^1 \leq g$. Moreover, if $x^3$ is any element of $X$ with $x^3 \nleq g$, then $g \geq x^3$. \(\square\)

3.2. Lemma. Let $X$ be a linearly ordered $\ell$-subgroup of $G$. Then $\lim_{\alpha} X$ is also a linearly ordered $\ell$-subgroup of $G$. 308
Proof. In view of 2.2, \( \lim X \) is an \( \ell \)-subgroup of \( G \). Hence it suffices to verify that whenever \( g_1 \) and \( g_2 \) are distinct elements of \( \lim X \), then \( g_1 \) and \( g_2 \) are comparable. In view of 3.1 there are ideals \( X_1 \) and \( X_2 \) of the linearly ordered set \( X \) such that

(i) \( X_1 \neq X \neq X_2 \);
(ii) \( x \leq g_1 \) if \( x \in X_1 \), and \( x > g_1 \) if \( x \in X \setminus X_1 \);
(iii) \( x \leq g_2 \) if \( x \in X_2 \), and \( x > g_2 \) if \( x \in X \setminus X_2 \).

The ideals \( X_1 \) and \( X_2 \) are comparable. Since \( g_1 \neq g_2 \), we obtain that \( X_1 \neq X_2 \). Thus without loss of generality we can suppose that \( X_1 \subset X_2 \). Hence there is \( z \in X_2 \setminus X_1 \). Then in view of (ii), \( z > g_1 \). Further, according to (iii), \( z \leq g_2 \). Therefore \( g_1 \leq g_2 \).

\[ \square \]

3.3. Lemma. Let \( X \) be a linearly ordered \( \ell \)-subgroup of \( G \). Then (1) holds.

Proof. Let \( (y_n) \) be a sequence in \( \lim X \), \( g \in G \) and \( y_n \to \alpha g \). Then in view of 3.1 and 3.2, \( g \) is comparable with all elements of \( \lim X \). Hence there exists a subsequence \( (y_n^1) \) of \( (y_n) \) such that either (i) \( y_n^1 \geq g \) for each \( n \in \mathbb{N} \), or (ii) \( y_n^1 \leq g \) for each \( n \in \mathbb{N} \). Suppose that (i) holds (in the case of (ii) the method is similar). If \( y_n^1 = g \) for some \( n \in \mathbb{N} \), then \( g \in \lim \alpha \). Thus it suffices to suppose that \( y_n^1 < g \) for each \( n \in \mathbb{N} \), and in this case we can assume without loss of generality that \( y_n^1 < y_{n+1}^1 \) for each \( n \in \mathbb{N} \).

Let \( n \in \mathbb{N} \). There exists a sequence \( (x_k^\alpha)_{k \in \mathbb{N}} \) in \( X \) such that \( x_k^\alpha \to \alpha y_n^1 \) (as \( k \to \infty \)). Hence there is \( m(n) \in \mathbb{N} \) such that

\[
y_n^1 < x_k^\alpha < y_{n+1}^1
\]

whenever \( k \geq m(n) \). Since \( y_{n+1}^1 \to \alpha g \) and \( y_n^1 \to \alpha g \) we obtain that

\[
x_{m(n)}^\alpha \to \alpha g
\]

and thus \( g \in \lim X \). Hence (1) is valid.

\[ \square \]

3.4. Lemma. Let \( L \) be a distributive lattice with the least element 0. Let \( A \) and \( B \) be sublattices of \( L \) such that

(i) \( 0 \in A \cap B \);
(ii) \( a \wedge b = 0 \text{ for each } a \in A \text{ and each } b \in B \);
(iii) for each \( g \in L \) there are \( a \in A \) and \( b \in B \) with \( g = a \vee b \).

309
Then the elements $a$, $b$ from (iii) are uniquely determined and the mapping $g \to (a, b)$ gives an isomorphism of $L$ onto the direct product $A \times B$.

The proof is routine, it will be omitted.

**3.5. Lemma.** Let $X$ be an $\ell$-subgroup of $G$ such that $X$ is a direct product of linearly ordered groups $X_1, X_2, \ldots, X_k$. Then the $\ell$-group \( \lim_a X \) is a direct product of linearly ordered groups \( \lim_a X_1, \lim_a X_2, \ldots, \lim_a X_k \).

**Proof.** We proceed by induction with respect to $k$. The case $k = 1$ is trivial. Suppose that $k > 1$ and that the assertion is valid for $k - 1$.

Without loss of generality we can assume that $X_i \neq \{0\}$ for $i = 1, 2, \ldots, k$. Put $Y_i = \lim_a X_i$ ($i = 1, 2, \ldots, k$). According to 3.2, all $Y_i$ are linearly ordered $\ell$-subgroups of $G$. Also, $\lim_a X = Y$ is an $\ell$-subgroup of $G$. In view of Theorem 2.3, [4] it suffices to verify that the lattice $Y^+$ is a direct product of lattices $Y_{i1}^+ \ldots Y_{ik}^+$.

Let $g \in Y^+$. In the same way as in the proof of 3.1 we can verify that $g$ fails to be an upper bound of the set $X^+$. For each $x \in X^+$ we have

$$x = x(X_1) \lor \ldots \lor x(X_k), \quad x(X_i) \geq 0 \quad (i = 1, 2, \ldots, k),$$

where $x(X_i)$ is the component of $x$ in $X_i$. Hence $g$ fails to be an upper bound of the set $X_1^+ \cup X_2^+ \cup \ldots \cup X_k^+$. Thus we can suppose that $g$ is not an upper bound of the set $X_k^+$. Therefore there is $x_0 \in X_k^+$ such that $x_0 \not\geq g$.

There is a sequence $(z_n)$ in $X$ such that $z_n \to_a g$. Put $z_n' = z_n \lor 0$. Then we have $z_n' \to_a g$ as well. Further,

$$z_n' \land x_0 = (z_n'(X_1) \lor z_n'(X_2) \lor \ldots \lor z_n'(X_k)) \land x_0 =$$

$$= z_n'(X_k) \land x_0 \in X_k$$

and $z_n'(X_k) \land x_0 \to_a g \land x_0$, whence $g \land x_0 \in \lim_a X_k \subseteq \lim_a X$.

Put $N_1 = \{ n \in \mathbb{N}: z_n'(X_k) \geq x_0 \}$. If the set $N_1$ is infinite, then there exists a subsequence $(z''_n)$ of $(z'_n)$ such that $z''_n \geq x_0$ for each $n \in \mathbb{N}$ and then we would have $g \geq x_0$, which is a contradiction. Hence the set $N_1$ is finite; thus there is a subsequence $(z''_n)$ of $(z'_n)$ such that $z''_n(X_k) < x_0$ for each $n \in \mathbb{N}$, whence

$$z''_n(X_k) \land x_0 = z''_n(X_k)$$

and then $z''_n(X_k) \to_a g \land x_0$. Therefore

$$z''_n - z''_n(X_k) = z''_n(X_1) + z''_n(X_2) + \ldots + z''_n(X_{k-1}) \to_a g - (g \land x_0).$$
Therefore by the induction hypothesis (since \( z_n^\#(X_1) + \ldots + z_n^\#(X_{k-1}) \) belongs to \( X_1 \times \ldots \times X_{k-1} \)) the element \( g - (g \land x_0) \) belongs to the direct product \( Y_1 \times Y_2 \times \ldots \times Y_{k-1} \). Since \( g - (g \land x_0) \geq 0 \) we obtain, moreover, that this element belongs to the direct product of lattices \( Y_1^+, \ldots, Y_{k-1}^+ \).

Let \( t \in Y_1^+ \times Y_2^+ \times \ldots \times Y_{k-1}^+ \). Then by the induction hypothesis, there is a sequence \( (t_n) \) in \( X_1^+ \times \ldots \times X_{k-1}^+ \) such that \( t_n \to t \). We have
\[
t_n \land z_n^\#(X_k) = 0 \quad \text{for each } n \in \mathbb{N},
\]
thus
\[
t \land (g \land x_0) = 0,
\]
\[
t + (g \land x_0) = t \lor (g \land x_0).\]
In particular,
\[
g = (g - (g \land x_0)) + (g \land x_0) = (g - (g \land x_0)) \lor (g \land x_0)
\]
with \( g - (g \land x_0) \in Y_1^+ \times \ldots \times Y_{k-1}^+ \) and \( g \land x_0 \in Y_k^+ \).

Hence in view of 3.4 we obtain that for the lattice \( Y^+ \) there exists a direct product decomposition
\[
Y^+ = Y_1^+ \times Y_2^+ \times \ldots \times Y_k^+.
\]
Now we apply again Theorem 2.3 of [4] concluding that the \( \ell \)-group \( Y \) has a direct product decomposition
\[
(2) \quad Y = Y_1 \times Y_2 \times \ldots \times Y_k.
\]

\[
\square
\]

3.6. Theorem. Let \( X \) be an \( \ell \)-subgroup of \( G \) such that \( X \) is a direct product of a finite number of linearly ordered groups. Then (1) holds.

Proof. We apply the notation as in the proof of 3.5 and similarly as in 3.5 we proceed by induction with respect to \( k \). The case \( k = 1 \) was dealt with in 3.3; let \( k > 1 \).

Since all \( Y_i \) are linearly ordered we can apply 3.5 to the relation (2) obtaining
\[
\lim_a Y = \lim_a Y_1 \times \lim_a Y_2 \times \ldots \times \lim_a Y_k.
\]
Since \( \lim_a Y = \lim_a X \) and \( \lim_a Y_i = \lim_a X_i \) \( (i = 1, 2, \ldots, k) \), by applying 3.3 we infer
\[
\lim_a X = Y_1 \times \ldots \times Y_k = \lim_a X.
\]

\[
\square
\]

311
4. The relation of partial order between radical classes

For a class $X$ of cl-groups we denote by

Sub$_cX$—the class of all cl-groups $(G, \alpha)$ having the property that there exist

$(H, \beta)$ in $X$ and $(H_i, \beta) \in c(H, \beta)$ such that $(G, \alpha)$ and $(H_i, \beta)$ are cl-isomorphic;

Join$_X$—the class of all cl-groups $(G, \alpha)$ having the property that there exist

$(H_i, \beta_i)$ in $X$ and $(G_i, \alpha) \in c(G, \alpha)$ $(i \in I)$ such that

a) for each $i \in I$, $(H_i, \beta_i)$ and $(G_i, \alpha)$ are cl-isomorphic, and

b) $(G, \alpha) = \bigvee_{i \in I}(G_i, \alpha)$.

4.1. Proposition. Let $X$ be a nonempty class of cl-groups. Then

a) Join$_{Sub_cX}$ $X$ is a radical class of cl-groups.

b) If $Y$ is a radical class of cl-groups and $X \subseteq Y$, then Join$_{Sub_cX}X \subseteq Y$.

Proof. Put Join$_{Sub_cX}X = Z$. We have to verify that $Z$ satisfies the conditions (i), (ii) and (iii) from 1.4. It is obvious that $Z$ is closed with respect to cl-isomorphisms. For each nonempty class $Z_1$ of cl-groups we have Join$_{Z_1} = JoinZ_1$, whence $Z$ satisfies the condition (iii) from 1.4.

Let $(G, \alpha) \in Z$ and $(G_i, \alpha) \in c(G, \alpha)$. Hence there exist $(H_i, \alpha_i) (i \in I)$ belonging to $Sub_cX \cap c(G, \alpha)$ such that

$$(G, \alpha) = \bigvee_{i \in I}(H_i, \alpha).$$

Then by applying 2.10

$$(G_1, \alpha) = (G_1, \alpha) \wedge^c (G, \alpha) = (G_1, \alpha) \wedge^c \left( \bigvee_{i \in I}(H_i, \alpha) \right)$$

$$= \bigvee_{i \in I}((G_1, \alpha) \wedge^c (H_i, \alpha)).$$

For each $i \in I$, the cl-group $(G_1, \alpha) \wedge^c (H_i, \alpha)$ belongs to Sub$_cSub_cX = Sub_cX$ and therefore $(G_1, \alpha)$ belongs to $Z$. Hence the condition (ii) from 1.3 is valid, which completes the proof of a).

Let $Y$ be a radical class of cl-groups and $X \subseteq Y$. Then Sub$_cX \subseteq Sub_cY = Y$ and Join$_{Sub_cX}X \subseteq JoinY = Y$. Thus b) is valid. \(\square\)

Let $Y_1$ and $Y_2$ be radical classes of cl-groups. We put $Y_1 \leq Y_2$ if $Y_1$ is a subclass of $Y_2$.

312
We denote by $Y_0$ the class of all cl-groups $(G, \alpha)$ such that $G$ is a one-element set. Then $Y_0$ is a radical class of cl-groups and for each radical class $Y$ of cl-groups we have $Y_0 \subseteq Y \subseteq \mathcal{G}_c$.

Let $G$ be an $\ell$-group. For a sequence $(x_n)$ in $G$ and for $x \in G$ we put $x_n \rightarrow_{\alpha(G)} x$ if there exists $m \in \mathbb{N}$ such that $x_n = x$ for each positive integer $n$ with $n \geq m$. Then $(G, \alpha(G))$ is a cl-group; $\alpha(G)$ is the discrete convergence on $G$.

If $X$ is a class of $\ell$-groups, then we put

$$\varphi(X) = \{(G; \alpha(G)) : G \in X\}.$$  

Then we obviously have

**4.2. Lemma.** If $X$ is a radical class of $\ell$-groups, then $\varphi(X)$ is a radical class of cl-groups. Moreover, if $X_1$ and $X_2$ are distinct radical classes of $\ell$-groups, then $\varphi(X_1) \neq \varphi(X_2)$.

Let $\mathcal{R}_a$ and $\mathcal{R}_c$ be the collection of all radical classes of $\ell$-groups or the collection of all radical classes of cl-groups, respectively. (Let us remark that in [5] the symbol $\mathcal{R}$ was used, but in [5] it was not assumed that the $\ell$-groups under consideration were abelian.)

There exists an injective mapping of the class of all infinite cardinals into $\mathcal{R}_a$ (this follows from the construction in [5], Section 3). Hence in view of 4.2, there exists an injective mapping of the class of all infinite cardinals into $\mathcal{R}_c$.

Suppose that $I$ is a nonempty class and that for each $i \in I$, $Y_i$ is a radical class of cl-groups. Put

$$Z_1 = \bigcap_{i \in I} Y_i.$$  

Then in view of 1.4, $Z_1$ is a radical class of cl-groups. We obviously have

$$Z_1 = \inf\{Y_i\}_{i \in I}.$$  

We express this fact by writing

$$Z_1 = \bigwedge_{i \in I} Y_i.$$  

Further, we put

$$Z_2 = \Join\{\bigcup_{i \in I} Y_i\}.$$  

Then 4.1 yields that the relation

$$Z_2 = \sup\{Y_i\}_{i \in I}$$  

313
is valid in $R_c$. We express this fact by writing
\[ Z_2 = \bigvee_{i \in I} Y_i. \]

We clearly have
\[ \text{Sub} \bigcup_{c \in I} Y_i = \bigcup_{i \in I} \text{Sub} Y_i. \]
Since each $Y_i$ is a radical class of $cl$-groups we obtain $\text{Sub}_c Y_i = Y_i$. Hence
\[ \bigvee_{i \in I} Y_i = \text{Join} \bigcup_{i \in I} Y_i. \]

4.3. Theorem. Let $\{Y_i\}_{i \in I}$ be as above and let $Y$ be a radical class of $cl$-groups. Then
\[ Y \land \left( \bigvee_{i \in I} Y_i \right) = \bigvee_{i \in I} (Y \land Y_i). \]

**Proof.** We have
\[ \bigvee_{i \in I} (Y \land Y_i) \leq Y \land \left( \bigvee_{i \in I} Y_i \right). \]
Let $(G, \alpha) \in Y \land \left( \bigvee_{i \in I} Y_i \right)$. Thus $(G, \alpha) \in Y$ and
\[ (G, \alpha) \in \text{Join} \bigcup_{i \in I} Y_i. \]
Then there exist $cl$-groups $(G_k, \alpha)$ ($k \in K$) such that, for each $k \in K$,
\[ (G_k, \alpha) \in c(G, \alpha) \cap \left( \bigcup_{i \in I} Y_i \right) \]
and
\[ (G, \alpha) = \bigvee_{k \in K} (G_k, \alpha). \]
Hence for each $k \in K$ there exists $i(k) \in I$ with $(G_k, \alpha) \in Y_{i(k)}$. Denote
\[ I_k = \{i(k) : k \in K\}. \]
Thus $(G_k, \alpha) \in Y \land Y_{i(k)}$ and
\[ (G, \alpha) \in \text{Join} \bigcup_{i \in I_k} (Y \land Y_{i(k)}) \leq \text{Join} \bigcup_{i \in I} (Y \land Y_i) = \bigvee_{i \in I} (Y \land Y_i). \]

$\square$

314
References


Author’s address: Ján Jakubík, Matematický ústav SAV, Grešíkova 6, 040 01 Košice, Slovakia.

315