THE CROSS-RATIO IN HJELMSLEV PLANES

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Abstract. The cross-ratio in Hjelmslev planes is defined. The cross-ratio in the Hjelmslev plane \( H(R) \) is independent of the choice of a coordinate system on a line.

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1. Introduction

A special local ring is a finite commutative local ring \( R \) the ideal \( I \) of divisors of zero of which is principal. Suppose that \( R \) is not a field and that the characteristic of \( R \) is odd. Denote the factor ring \( R/I \) by the symbol \( \overline{R} \). Further denote the set of all regular elements of \( R \) by the symbol \( R^* \), thus \( R^* = R - I \).

Definition 1.1. A projective Hjelmslev plane (we will denote it by \( H(R) \)) over \( R \) is an incidence structure \( H(R) = (B; \mathcal{P}; \mathcal{T}) \) defined in the following way:

- the elements of \( B \)—the points of \( H(R) \) are classes of ordered triples \( (\lambda x_1; \lambda x_2; \lambda x_3) \) where \( \lambda \in R^* \), \( x_1, x_2, x_3 \in R \) and at least one \( x_i \) is regular;
- the elements of \( \mathcal{P} \)—the lines of \( H(R) \) are classes of ordered triples \( (\alpha a_1; \alpha a_2; \alpha a_3) \) where \( \alpha \in R^* \), \( a_1, a_2, a_3 \in R \) and at least one \( a_i \) is regular.

A point \( X = [x_1; x_2; x_3] \) is incident to the line \( a = [a_1; a_2; a_3] \) if and only if

\[
(1.1) \quad a_1 x_1 + a_2 x_2 + a_3 x_3 = 0.
\]

Remark 1.1. The canonical homomorphism \( \Phi: R \to R/I = \overline{R} \) induces a homomorphism of \( H(R) \) onto the projective plane \( \pi(\overline{R}) \).
We will call the points \( X, Y \in H(R) \) neighbouring if \( X = Y \) where \( \Phi(X) = X, \Phi(Y) = Y \). Similarly we will call points \( X, Y \in H(R) \) substantially different if \( X \neq Y \). Two lines are neighbouring if there are points \( A_1, A_2 \in \mathcal{L}, A_1 \neq A_2 \) such that \( A_1 \mathcal{L} a, b \) and \( A_2 \mathcal{L} a, b \). Let \( X \) be a subset of the \( R \)-modul \( M \) and let \( j: X \rightarrow M \) be an insertion of the subset \( X \) into \( M \). Then \( M(R) \) is called the free modul over \( X \) if for an arbitrary function \( f: X \rightarrow A \) into the \( R \)-modul \( A \) there is exactly one linear mapping \( t: M(R) \rightarrow A \) such that \( t \circ j = f \).

Remark 1.2. The analytic model of the Hjelmslev plane, introduced by definition 1.1 is really a free modul over \( R \) with a factorization defined in the following way: triples \( (x_1; x_2; x_3) \) and \( (x'_1; x'_2; x'_3) \) are identical if there is \( \lambda \in \mathbb{R}^+ \) such that \( x'_i = \lambda x_i \) for \( i = 1, 2, 3 \) and we do not consider the zero triple.

2. The construction and proof of theorem

Definition 2.1. A coordinate system in \( H(R) \) is an ordered quadruple of points \( E_1, E_2, E_3, E_4 \) such that the points \( \overline{E_1}, \overline{E_2}, \overline{E_3}, \overline{E_4} \) generate a coordinate system in \( \pi(R) \).

If a point \( X = [x_1; x_2; x_3] \) is given by the vector \( x = (x_1; x_2; x_3) \), we write \( X = (x) \).

Lemma 2.1. Let \( M(R) \) be a free modul over \( R \) and let \( e_1, e_2, e_3 \) be a basis of \( M(R) \). Then the points \( E_1 = (e_1), E_2 = (e_2), E_3 = (e_3), E_4 = (e_1 + e_2 + e_3) \) generate the coordinate system in the Hjelmslev plane \( H(R) \) corresponding to the modul \( M(R) \).

Proof. It is necessary to prove that the points \( \overline{E_1}, \overline{E_2}, \overline{E_3}, \overline{E_4} \) generate a coordinate system in \( \pi(R) \). Obviously \( \overline{e_1}, \overline{e_2}, \overline{e_3} \) form a basis of a vector space over \( R \) and thus the vectors \( \overline{e_1}, \overline{e_2}, \overline{e_3} \) are linearly independent. It follows that the points \( \overline{E_1} = (\overline{e_1}), \overline{E_2} = (\overline{e_2}), \overline{E_3} = (\overline{e_3}) \) and \( \overline{E_4} = (\overline{e_1} + \overline{e_2} + \overline{e_3}) \) are not on a unique line. □

Conversely, we have

Lemma 2.2. Let \( E_1, E_2, E_3, E_4 \) be a coordinate system in \( H(R) \). Then there is a basis of the modul \( M(R) \) such that \( (e_1) = E_1, (e_2) = E_2, (e_3) = E_3, (e_1 + e_2 + e_3) = E_4 \).

Proof. Let \( E_1 = (b_1), E_2 = (b_2), E_3 = (b_3) \) and \( E_4 = (b_4) \). Because \( \{b_1, b_2, b_3\} \) is a basis of \( M(R) \) the vector \( b_4 \) can be expressed in the form

\[
b_4 = \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3.
\]
If we denote $c_1 = \beta_1 b_1$, $c_2 = \beta_2 b_2$, $c_3 = \beta_3 b_3$ then $c_1, c_2, c_3$ are the vectors from the statement of the lemma.

Let $E_1, E_2, E_3, E_4$ and $E'_1, E'_2, E'_3, E'_4$ be coordinate systems in $H(R)$. If $c_1, c_2, c_3$ and $c'_1, c'_2, c'_3$ are the corresponding bases of the modul $M(R)$ then there is a regular matrix $A = [a_{ij}]$ such that

$$c'_i = \sum_j a_{ij} c_j, \quad i = 1, 2, 3.$$  

Let $X_E = [x_1; x_2; x_3]$, $X'_E = [x'_1; x'_2; x'_3]$. Then

$$x = \sum_i x'_i c'_i = \sum_i x'_i \sum_j a_{ij} c_j = \sum_j \left( \sum_i x'_i a_{ij} \right) c_j = \sum_j x_j c_j.$$  

Comparing the two identities, we get

$$x_j = \sum_i x'_i a_{ij}. \quad (2.1)$$  

The relation (2.1) can be written also in the form

$$X_E = X'_E A, \quad X'_E = X_E A^{-1}. \quad (2.2)$$  

Let an invertible matrix $A$ and a coordinate system $E_1, E_2, E_3, E_4$ be given, then points $E'_1, E'_2, E'_3, E'_4$ generate a coordinate system and the corresponding vectors of the point $X \in H(R)$ satisfy

$$X_E = X'_E A.$$  

Let the special local ring $R$ be given. We introduce a set $\Omega$ by

$$\Omega \cap R = \emptyset, \quad |\Omega| = |I|. \quad (2.3)$$  

Thus there is a bijective mapping $\omega$ such that

$$\omega: I \rightarrow \Omega, \quad \omega: i \rightarrow \omega_i = \omega(i), \quad i \in I \quad (2.4)$$  

where $\omega_i$ are “inverse” elements of elements $i \in I$, thus $\omega_i \sim 1/i$. $\Omega$ is the set of “infinities” corresponding to singular elements. Define an extension of the canonical homomorphism $\Phi$ to the set $R \cup \Omega$, let us put

$$\Phi(\Omega) = \infty. \quad (2.5)$$  

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Let \( A, B, E \) be three substantially different points generating a coordinate system on a line. Then every point \( X \) of this line can be expressed uniquely (the single-valuedness guarantees the point \( E \)) in the form

\[
X = sA + tB
\]

and hence the point \( X = [s; t] \) is determined by the pair \( (s, t) \).

On the line with the coordinate system \( A, B, E \) let us have points \( P_1, P_2, P_3, P_4 \) where \( P_1 = s_1 A + t_1 B \) thus \( P_1 = [s_1; t_1] \).

**Definition 2.2.** The cross-ratio of an ordered quadruple of points \( P_1, P_2, P_3, P_4 \) on a line in \( \mathcal{H}(\mathbb{R}) \), of which at least three are substantially different is an element \( (P_1 P_2 P_3 P_4) \in R \cup \Omega \) which is defined by relations

\[
(P_1 P_2 P_3 P_4) = \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \\ s_3 & t_3 \\ s_4 & t_4 \end{vmatrix}
\]

if points \( P_1 P_4 \) and \( P_2 P_3 \) are substantially different,

\[
(P_1 P_2 P_3 P_4) = \omega(P_1 P_2 P_3 P_4)
\]

if points \( P_1, P_4 \) and \( P_2, P_3 \) are neighbouring. Suppose that points \( P_1, P_3 \) and \( P_2, P_4 \) are substantially different.

**Remark.** If \( \mathcal{H} \) is a field, \( I = \{0\} \) then Definition 2.2 is the same as the definition of the cross-ratio in a projective plane.

**Theorem 2.3.** The cross-ratio introduced by relations 2.7 and 2.8 is independent of the choice of a coordinate system on the line.

**Proof.** Let a line \( p \in \mathcal{H}(\mathbb{R}) \) be given and on this line let us have coordinate systems \( A, B, E \) and \( A', B', E' \). Let \( P_1, P_2, P_3, P_4 \) be points on the given line \( p \) whose the cross-ratio we want to investigate. There is obviously a linear transformation which maps the points \( A, B \) to the points \( A', B' \) on \( p \). We want to verify that the cross-ratio is independent of the choice of the coordinate points on the line. Thus

\[
(P_1 P_2 P_3 P_4)_{A'B'} = (P_1 P_2 P_3 P_4)_{A'B'}.
\]

We have

\[
A' = a_1 A + a_2 B \\
B' = b_1 A + b_2 B
\]

We have
and thus
\[ P_1 = s_1A' + t_1B' \]
and after a substitution we get
\[ P_1 = (s_1'a_1 + t_1'b_1)A + (s_1'a_2 + t_1'b_2)B = s_1A + t_1B, \quad i = 1, 2, 3, 4. \]

By direct calculation we obtain \((P_1P_2, P_3P_4)_{AB} = (P_1P_2, P_3P_4)_{AB'}\) which was to be proved. \(\square\)

References


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